An Improved Upper Bound on the Density of Universal Random Graphs

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Abstract. We give a polynomial time randomized algorithm that, on receiving as input a pair (H, G) of n-vertex graphs, searches for an embedding of H into G. If H has bounded maximum degree and G is suitably dense and pseudorandom, then the algorithm succeeds with high probability. Our algorithm proves that, for every integer $d \geq 3$ and suitable constant $C = C_d$, as $n \to \infty$, asymptotically almost all graphs with n vertices and $\lfloor Cn^{2-1/d} \log^{1/d} n \rfloor$ edges contain as subgraphs all graphs with n vertices and maximum degree at most d.

1 Introduction

Given graphs H and G, an *embedding* of H into G is an injective edge-preserving map $f : V(H) \to V(G)$, that is, such that, for every $e = \{u, v\} \in E(H)$, we have $f(e) = \{f(u), f(v)\} \in E(G)$. We shall say that a graph H *is contained in* G *as a subgraph* if there is an embedding of H into G. Given a family of graphs $\mathcal{H}$, we say that G is *universal with respect to* $\mathcal{H}$, or $\mathcal{H}$*-universal*, if every $H \in \mathcal{H}$ is contained in G as a subgraph.

The construction of sparse universal graphs for various graph families has received a considerable amount of attention; see, e.g., [1,3,4,5,6,7,8,10] and the references therein. One is particularly interested in (*almost*) *tight* $\mathcal{H}$-universal graphs, i.e., graphs whose number of vertices is (*almost*) *equal* to $\max_{H \in \mathcal{H}} |V(H)|$.

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Let $d \in \mathbb{N}$ be a fixed constant and let $\mathcal{H}(n,d) = \{H \subset K_n : \Delta(H) \leq d\}$ denote the class of (pairwise non-isomorphic) n-vertex graphs with maximum degree bounded by d and $\mathcal{H}(n,n;d) = \{H \subset K_{n,n} : \Delta(H) \leq d\}$ be the corresponding class for balanced bipartite graphs.

By counting all unlabeled d-regular graphs on n vertices one can easily show that every $\mathcal{H}(n,d)$-universal graph must have

$$\Omega(n^{2-2/d}) \tag{1}$$

edges (see [3] for details). This lower bound was almost matched by a construction from [4], which was subsequently improved in [1] and [2]. Those constructions were designed to achieve a nearly optimal bound and as such they did not resemble a "typical" graph with the same number of edges. To pursue this direction, in [3], the $\mathcal{H}(n,d)$-universality of the random graphs $G_{n,p}$ was investigated.

For random graphs a slightly better lower bound than (1) is known. Indeed, any $\mathcal{H}(n,d)$-universal graph must contain as a subgraph the union of $\lfloor n/(d+1) \rfloor$ vertex-disjoint copies of K_{d+1}, and, in particular, all but at most d vertices must each belong to a copy of K_{d+1}. Therefore, recalling the threshold for the latter property (see, e.g., [14, Theorem 3.22(i)]), we conclude that the expected number of edges needed for the $\mathcal{H}(n,d)$-universality of $G_{n,p}$ must be

$$\Omega\left(n^{2-2/(d+1)}(\log n)^{1/\binom{d+1}{2}}\right), \tag{2}$$

a quantity bigger than (1).

We say that $G_{n,p}$ possesses a property $\mathcal{P}$ *asymptotically almost surely* (**a.a.s.**) if $\mathbf{P}[G_{n,p} \in \mathcal{P}] = 1 - o(1)$. We write $G_{n,n,p}$ for the random balanced bipartite graph on $2n$ vertices and edge probability p. In [3], it was proved that for a sufficiently large constant C:

A (almost tight universality) The random graph $G_{(1+\varepsilon)n,p}$ is **a.a.s.** $\mathcal{H}(n,d)$-universal if $p = Cn^{-1/d}\log^{1/d} n$;

B (tight bipartite universality) The random bipartite graph $G_{n,n,p}$ is **a.a.s.** $\mathcal{H}(n,n,d)$-universal if $p = Cn^{-1/2d}\log^{1/2d} n$.

Note that (A) above deals with embeddings of n-vertex graphs into random graphs with slightly larger vertex sets, which makes the embedding somewhat easier. On the other hand, (B) above deals with tight universality at the cost of requiring the graphs to be bipartite and with a less satisfactory bound.

Those results were improved and extended by the authors in [9,11], where it was shown that $G_{n,n,p}$ is **a.a.s.** $\mathcal{H}(n,n,d)$-universal if $p = Cn^{-1/d}\log^{1/d} n$, and $G_{n,p}$ is **a.a.s.** $\mathcal{H}(n,d)$-universal if $p = Cn^{-1/2d}\log^{1/2d} n$. In this paper, making use of an additional randomization step in the embedding algorithm involved, we improve the latter result, establishing a density threshold for the $\mathcal{H}(n,d)$-universality of $G_{n,p}$ that matches the best previous bounds for both the bipartite tight universality and the almost tight universality in the general case.

Theorem 1. *Let $d \geq 3$ be fixed and suppose $p = p(n) = Cn^{-1/d} \log^{1/d} n$ for some sufficiently large constant C. Then the random graph $G_{n,p}$ is* ***a.a.s.*** *$\mathcal{H}(n,d)$-universal.*

Standard methods let us derive from Theorem 1 the $\mathcal{H}(n,d)$-universality of almost all n-vertex graphs with $M = \lfloor Cn^{2-1/d} \log^{1/d} n \rfloor$ edges. Observe that there is still a gap between the lower bound (2) and the upper bound given by Theorem 1. We remark that $n^{-1/d} \log^{1/d} n$ is a natural barrier for the problem considered here, as this is roughly the point where every d-tuple of vertices of $G_{n,p}$ shares a common neighbor.

Remark 1. In Theorem 1 we assume that $d \geq 3$ since for $d = 2$ our proof would require a few modifications. On the other hand, we feel that the true bound for $d = 2$ is much lower. Possibly as low as (2), which, as proved by Johansson, Kahn, and Vu [16], is also the threshold for triangle-factors in $G_{n,p}$. The case $d = 2$ will be dealt with elsewhere. We assume that $d \geq 3$ throughout.

This paper is organized as follows. In the next section we describe a randomized algorithm that seeks, for any $H \in \mathcal{H}(n,d)$ and any n-vertex graph G, an embedding $f\colon V(H) \to V(G)$. Crucially, at the beginning of our algorithm, a collection of pairwise vertex-disjoint d-cliques is sampled from a certain subset of vertices of G, uniformly at random. This randomization allows us to verify a Hall-type condition that we use to embed the final group of vertices in the algorithm. This is formally stated in Lemma 4.

In Section 4, we prove that our algorithm succeeds with high probability for every $H \in \mathcal{H}(n,d)$ when run on $G_{n,p}$, as long as $p = Cn^{-1/d} \log^{1/d} n$ and $C = C_d$ is a large enough constant. Several relevant properties of $G_{n,p}$ for such a p are singled out in Section 3.

We shall use the following notation throughout. For $v \in V = V(G)$, let $G(v)$ denote the neighborhood of the vertex v in G. For $T \subset V$, let

$$G(T) = \{v \in V \setminus T\colon G(v) \cap T \neq \emptyset\} = \bigcup_{u \in T} G(u) \setminus T$$

denote the neighborhood of the set T in G in $V \setminus T$. For $T \subset V$, let $G[T]$ denote the subgraph of G induced by T. If J is a graph, when there is no danger of confusion, we write J for its edge set as well. For tidiness, we omit floor and ceiling signs whenever they are not important.

2 The Embedding

Let

$$\varepsilon = \varepsilon(d) = \frac{1}{100d^4}. \tag{3}$$

In what follows, when necessary, we tacitly assume that n is larger than a suitably large constant $n_0 = n_0(d)$. Given an n-vertex graph G, set $V := V(G)$ and let

$$V = V_0 \cup R_1 \cup \cdots \cup R_{d^2+2}, \quad \text{where } |R_i| = \varepsilon n \text{ for all } i, \tag{4}$$

be a fixed partition of V.

Without loss of generality, we shall assume that H is a maximal graph from $\mathcal{H}(n,d)$ in the sense that adding any edge to H increases its maximum degree beyond d. Since in such a graph the vertices with degrees smaller than d must form a clique, there are at most d of them.

We set $X := V(H)$, and fix an integer $t = \tau n = \tau|V|$, where

$$\tau = 2\varepsilon = \frac{1}{50d^4}. \tag{5}$$

In the embedding algorithm, we shall use the following preprocessing procedure of H.

The preprocessing of H: Select vertices $x_1, \dots, x_t \in X$ in such a way that they all have degree d and form a 3-*independent set* in H, that is, every pair of distinct vertices x_i, x_j is at distance at least four. (Owing to our choice of t, we may find these t vertices by a simple greedy algorithm.) Let $S_i = H(x_i)$ for all $i = 1, \dots, t$, and set

$$X_0 := \bigcup_{j=1}^{t} S_j.$$

Note that, by the 3-independence of the x_i $(1 \le i \le t)$, for all $i \ne j$ not only $S_i \cap S_j = \emptyset$, but also *there is no edge between* S_i *and* S_j *in* H.

Next, consider the square H^2 of the graph H obtained from H by adding edges between all pairs of vertices at distance two. Since the maximum degree of H^2 is at most d^2, by the Hajnal–Szemerédi Theorem [12] applied to H^2, there is a partition $X = X_1' \cup X_2' \cup \dots \cup X_{d^2+1}'$, such that all the sets X_i', $1 \le i \le d^2+1$, are independent in H^2, and thus 2-independent in H, and have roughly the same size, that is, $\big||X_i'| - |X_j'|\big| \le 1$ for all i and j. (In fact, we apply here an algorithmic version from [17] (see also [18]), which yields a polynomial time algorithm.) Finally, set

$$X_i = X_i' \setminus \{x_1, \dots, x_t\} \setminus X_0, \quad i = 1, \dots, d^2+1,$$

and $X_{d^2+2} = \{x_1, \dots, x_t\}$. Hence, we obtain the partition

$$X = X_0 \cup X_1 \cup \dots \cup X_{d^2+2}, \tag{6}$$

where, for $i = 1, \dots, d^2+1$, the sets X_i are 2-independent and

$$|X_i| \ge \frac{n}{d^2+1} - 1 - t(d+1) \ge \frac{n}{2d^2} > t, \tag{7}$$

while X_{d^2+2} is 3-independent, $|X_{d^2+2}| = t$, and X_0 is the (disjoint) union of the d-element neighborhoods of the vertices in X_{d^2+2}. (See Figure 1 for an illustration of this partition.) The numbering of the sets $X_0, \dots, X_{d^2+2}$ corresponds to the order in which these sets will be embedded into G by the embedding algorithm.

Another building block of our embedding algorithm is a procedure that, given a partial embedding f_{i-1} of $H[X_0 \cup \dots \cup X_{i-1}]$ into G, constructs an auxiliary graph A_i making explicit which vertices of G are candidates for becoming images of the vertices in X_i.

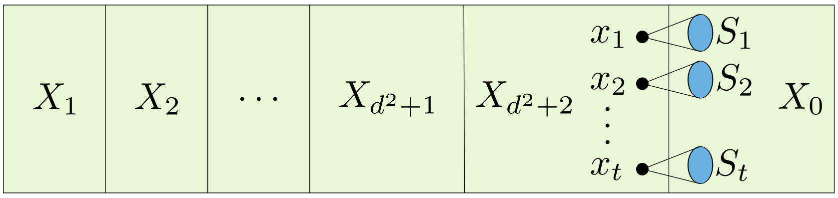

Fig. 1. The partition of $V(H)$

The auxiliary graph A_i: For $i = 1, \dots, d^2 + 2$ and a partial embedding $f_{i-1} : X_0 \cup \cdots \cup X_{i-1} \to V$, let A_i be the bipartite graph with vertex classes X_i and

$$W_i := V \setminus \operatorname{im}(f_{i-1}) \setminus \bigcup_{i<j\le d^2+2} R_j$$

(the R_j are as in (4)) and the edge set

$$\big\{(x, v) \in X_i \times W_i : f_{i-1}\big(H(x)\big) \subset G(v)\big\}. \tag{8}$$

Observe that $A_i(x)$ is the set of all the vertices $v \in W_i$ for which $x \mapsto v$ is a valid extension of the embedding f_{i-1}, while $A_i(v)$ is the set of all the vertices $x \in X_i$ for which v is a valid image.

Since the set X_i is independent, X_i can be embedded 'at once'; that is, it suffices to specify a matching in A_i saturating X_i. (The 2-independence of the X_is will only be used in the analysis of the algorithm.) Note that $|W_{d^2+2}| = |X_{d^2+2}|$, while for $1 \le i \le d^2 + 1$ the set W_i is noticeably bigger than the set X_i. Indeed,

$$\begin{aligned} |W_i| &= n - \sum_{0\le j<i} |X_j| - \sum_{i<j\le d^2+2} |R_j| \\ &= |X_i| + \sum_{i<j\le d^2+2} (|X_j| - |R_j|) \ge |X_i| + \varepsilon n. \end{aligned} \tag{9}$$

The embedding will proceed in $d^2 + 2$ rounds, split into three phases:

Phase 1: The sets $S_1, \dots, S_t$ are mapped *randomly* onto disjoint cliques of $G[V_0]$.
Phase 2: The sets X_i $(1 \le i \le d^2 + 1)$ are embedded, one by one, into the W_i.
Phase 3: The set X_{d^2+2} is mapped one-to-one onto W_{d^2+2} (the set of t remaining vertices of G).

A potential problem for our proposed embedding scheme is that the candidate set for a given vertex $x \in X = V(H)$ may be depleted before we have a chance to embed x. If that happens, there is no hope to complete the embedding. Similarly, a vertex $v \in V = V(G)$ may lose all of its neighbors in the auxiliary graphs A_i as a result of an unfortunate sequence of extensions. In other words, v can be excluded from all candidate sets and thus cannot be used in the embedding.

Since we have to use all vertices of V in the embedding, we must avoid this event as well. Our algorithm incorporates two devices that help us address these problems.

Buffer vertices in G (used in Phases 2 and 3). We shall make sure that, for each $i = 1, \ldots, d^2 + 2$, $\mathrm{im}(f_{i-1}) \cap R_i = \emptyset$ (see Line 5 of Algorithm 1). This way, R_i will be reserved as a *buffer* to help us embed the set X_i, provided the sets R_i will satisfy certain properties in G; see Section 3.

Buffer vertices in H (used in Phase 3). Since the neighborhoods S_j of the vertices x_j from X_{d^2+2} are embedded during Phase 1, for any given $v \in V$, *the vertices in X_{d^2+2} that can be mapped onto v* remain the same throughout Phase 2 (up until v is in fact used by the embedding). This will help us ensure the existence of a perfect matching in A_{d^2+2} in Phase 3, provided the random choices of $f(S_j)$ satisfy certain properties; see Lemma 4.

Now we present our embedding algorithm (see Algorithm 1).

Algorithm 1. The embedding algorithm

Input : A graph H with n vertices and $\Delta(H) \leq d$ and a graph G together with a vertex partition (4).

Output: An embedding $f\colon V(H) \to V(G)$ (or the algorithm fails).

`// Phase 1`

1 Preprocess H, obtaining a partition $X = X_0 \cup \cdots \cup X_{d^2+2}$ as in (6), where $X_0 = S_1 \cup \cdots \cup S_t$, $X_{d^2+2} = \{x_1, \ldots, x_t\}$, and $H(x_j) = S_j$, $j = 1, \ldots, t$.

2 Randomly select from V_0 a sequence of pairwise disjoint d-element sets $T_1, \ldots, T_t$ such that, for each $i = 1, \ldots, t$, $G[T_i]$ is a clique, with all such sequences equiprobable.

3 Define a map $f_0\colon X_0 \to \bigcup_{i=1}^{t} T_i$ in such a way that $f_0(S_i) = T_i$ for each $i = 1, \ldots t$.

`// Phase 2`

4 **for** $i = 1, 2, \ldots, d^2 + 1$ **do**

5   Set $W_i = V \setminus \mathrm{im}(f_{i-1}) \setminus \bigcup_{i<j\leq d^2+2} R_j$;

6   Construct the auxiliary bipartite graph A_i between the sets X_i and W_i, and find therein a matching M_i of size $|M_i| = |X_i|$.

7   Define the extension f_i of f_{i-1} by setting $f_i(x) = v$ for all $x \in X_i$, where $(x, v) \in M_i$, and $f_i(x) = f_{i-1}(x)$ for all $x \in X_0 \cup \cdots \cup X_{i-1}$.

`// Phase 3`

8 Set $W_{d^2+2} = V \setminus \mathrm{im}(f_{d^2+1})$. Note that $R_{d^2+2} \subset W_{d^2+2}$.

9 Construct the auxiliary bipartite graph A_{d^2+2} between sets X_{d^2+2} and W_{d^2+2}, and find therein a perfect matching M_{d^2+2}.

10 Define the output embedding f by setting $f(x) = v$ for all $x \in X_{d^2+2}$, where $(x, v) \in M_{d^2+2}$, and $f(x) = f_{d^2+1}(x)$ for all $x \in X \setminus X_{d^2+2}$.

Algorithm 1 finds an embedding of H into G as long as it is successful on Lines 2, 6 and 9. The sets S_i are embedded into V_0 by uniformly sampling a sequence of pairwise disjoint d-subsets $T_1, \ldots, T_t \subset V_0$ with every T_i inducing a clique in G. Thus, one (trivial) necessary condition for the success of the algorithm is that G should contain at least t disjoint cliques K_d. Notice that the map f_0 is an embedding, since the edges within S_i are clearly preserved ($G[T_i]$ is a clique), while $e_H(S_i, S_j) = 0$ holds for all $j \neq i$ by construction.

Two more demanding conditions are that the auxiliary bipartite graphs A_i from Lines 6 and 9 should possess the required matchings. Superficially, we could have combined the last two phases by including round d^2+2 into the loop, however we chose not to do so, because of the much more involved analysis of Phase 3. Indeed, it is a great deal harder to prove the existence of a perfect matching in the balanced bipartite graph A_{d^2+2} than to prove the existence of a matching saturating the X_i side of A_i ($1 \leq i < d^2+2$), because its W_i side is noticeably bigger (see (9)).

It is worth pointing out that the success of Phase 3 relies entirely on the (random) outcome of Phase 1. The algorithm's goal in Phase 3 is to find a perfect matching in the auxiliary bipartite graph A_{d^2+2} (which has vertex classes X_{d^2+2} and W_{d^2+2}). Recall that the neighborhoods $S_j = H(x_j)$ of the vertices $x_j \in X_{d^2+2}$ are completely embedded in Phase 1. Since f_{d^2+1} is an extension of f_0, for each $x_j \in X_{d^2+2}$ we have $f_{d^2+1}(S_j) = f_0(S_j) = T_j$. This implies that, for every $v \in W_{d^2+2}$, by definition, $(x_j, v) \in A_{d^2+2}$ if and only if $T_j \subset G(v)$. Let $\widetilde{A}_1$ be the bipartite graph with vertex classes $V(H) \setminus X_0$ and $V(G) \setminus \mathrm{im}(f_0)$ with (x, v) an edge in $\widetilde{A}_1$ if and only if $f_0(H(x)) \subset G(v)$. Then $A_1 = \widetilde{A}_1[X_1 \cup W_1]$ and, crucially,

$$A_{d^2+2} = \widetilde{A}_1[X_{d^2+2} \cup W_{d^2+2}]. \tag{10}$$

This observation will be utilized in the analysis of Algorithm 1 in Section 4.

3 Random Graphs

In this section we show that the random graph $G_{n,p}$ with $p = p(n)$ as in Theorem 1 **a.a.s.** satisfies several properties with respect to the distribution of edges and cliques. These properties are singled out in order to guarantee jointly the tight $\mathcal{H}(n,d)$-universality of $G_{n,p}$. More specifically, in Section 4 we shall show that Algorithm 1, which is a randomized algorithm, is successful with high probability on all pairs of input graphs (H, G), where $H \in \mathcal{H}(n,d)$ and G satisfies all these properties. But first we need some more notation.

- Given a graph G and a subset of vertices $U \subset V = V(G)$, denote by

$$\binom{U}{K_d}$$

  the family of all d-element sets $T \subset V$ such that the subgraph of G induced by T is complete, that is, $G[T] \cong K_d$.

– Given a family $\mathcal{X} = \{J_1, \ldots, J_r\}$ of pairwise disjoint subsets of V and a set $U \subset V$, let $B = B(\mathcal{X}, U)$ be the bipartite graph with vertex classes $\mathcal{X}$ and $U_{\mathcal{X}} := U \setminus \bigcup_{i=1}^{r} J_i$, with the edge (J_i, v) included in B whenever $G(v) \supset J_i$. Furthermore, let

$$\alpha(\mathcal{X}, U) = \big|\{v \in U_{\mathcal{X}} : \deg_B(v) \geq 1\}\big|.$$

  If all the sets J_i are singletons, then we write $B(Y, U)$ instead of $B(\mathcal{X}, U)$ and $\alpha(Y, U)$ instead of $\alpha(\mathcal{X}, U)$, where $Y = \bigcup J_i$. Note that in this special case $\alpha(Y, U) = |G(Y) \cap U|$.
– We write $a = (1 \pm \delta)b$ whenever $(1-\delta)b \leq a \leq (1+\delta)b$.
– Set

$$\omega = C \log n. \tag{11}$$

Let $\varepsilon = \varepsilon(d) > 0$ be as in (3). Set $V = [n]$ and fix a partition

$$V = V_0 \cup R_1 \cup \cdots \cup R_{d^2+2}$$

satisfying (4). By (3),

$$|V_0| = n - (d^2+2)\varepsilon n \geq \frac{3n}{4}. \tag{12}$$

Lemma 1 below summarizes several properties of $G_{n,p}$ that are important for us. Besides the use of standard Chernoff bounds, the proof of Lemma 1 involves the application of certain large deviation bounds for subgraph counts (see [13] and [15]); we omit the details.

Lemma 1. *For every $\delta > 0$, there exists $C > 0$ such that the random graph $G = G_{n,p}$ with $p \geq Cn^{-1/d} \log^{1/d} n$* ***a.a.s.*** *satisfies Properties* (**I**)–(**V**) *below.*

(**I**) (*a*) *For all $y \in V$,*

$$|G(y) \cap V_0| = (1+o(1))p|V_0|.$$

(*b*) *For all $y \neq y' \in V$,*

$$|G(y) \cap G(y') \cap V_0| = (1+o(1))p^2|V_0|.$$

(**II**) (*a*) *For all $Y \subset V$ with $|Y| \leq \delta p^{-1}$,*

$$|G(Y) \cap V_0| = (1 \pm 2\delta)p\,|Y|\,|V_0|. \tag{13}$$

(*b*) *For all $Y \subset V$ with $|Y| \geq \omega p^{-1}$ and $U \subset V \setminus Y$ with $|U| \geq \omega p^{-1}$,*

$$|B(Y, U)| = (1 \pm \delta)p\,|Y|\,|U|. \tag{14}$$

(**III**) (*a*) *For all $r \leq \delta p^{-d}$, every family $\mathcal{X} = \{J_1, \ldots, J_r\}$ of pairwise disjoint d-subsets of V, and for every set $U \in \{V_0, R_1, \ldots, R_{d^2+2}, V\}$, we have*

$$\alpha(\mathcal{X}, U) = (1 \pm \delta)p^d r\,|U|. \tag{15}$$

(*b*) *For all* $r \geq \omega p^{-d}$*, every family* $\mathcal{X} = \{J_1, \ldots, J_r\}$ *of pairwise disjoint* d*-subsets of* V*, and* $U \subset V \setminus \bigcup_{i=1}^{r} J_i$ *with* $|U| \geq \omega p^{-d}$*,*

$$|B(\mathcal{X}, U)| = (1 \pm \delta) p^d r\, |U|. \tag{16}$$

(**IV**) *We have*

$$\left| \binom{U}{K_d} \right| = (1 \pm \delta) p^{\binom{d}{2}} \binom{|U|}{d} \tag{17}$$

for all $U \subset V$ *such that*

(*a*) $U \subset G(v)$ *for some* $v \in V$ *and* $|U| \geq pn/3$*, or*

(*b*) $U = G(u) \cap G(v)$ *for some distinct* u *and* $v \in V$*, or*

(*c*) $|U| \geq n/4$*.*

(**V**) *For all* $v \in V_0$*, the number of* d*-cliques in* $G[V_0]$ *containing* v *is*

$$(1 \pm \delta) \frac{d}{|V_0|} \left| \binom{V_0}{K_d} \right|.$$

4 The Analysis of Algorithm 1

In this section we derive Lemma 2 below, which together with Lemma 1, implies Theorem 1.

Lemma 2. *Let* $d \geq 3$ *be fixed. Let* ε *and* τ *be as in* (3) *and* (5)*, set* $\delta = 0.01$ *and suppose* $C \geq C(d)$ *is large enough. Then, for any* $\eta > 0$*, there is* n_0 *such that the following holds for all* $n \geq n_0$*. Let a graph* G *on the vertex set* $V = [n]$ *and a partition* $V = V_0 \cup R_1 \cup \cdots \cup R_{d^2+2}$ *as in* (4) *satisfy Properties* (**I**)–(**V**) *from Lemma 1 with* δ *and* C *as above. Furthermore, let* $H \in \mathcal{H}(n, d)$ *be given. Then, with probability at least* $1 - \eta$*, Algorithm 1 is successful on input* (H, G)*, that is, it outputs an embedding of* H *into* G*.*

We stress that the probability specified in Lemma 2 refers solely to the random choice of $T_1, \ldots, T_t$ on Line 2 in Algorithm 1. Note that, in particular, it follows that any graph G satisfying the hypotheses in Lemma 2 is $\mathcal{H}(n, d)$-universal.

As mentioned before, Algorithm 1 is successful if it does not terminate at Lines 2, 6, or 9. To execute Line 2 we need to have at least t disjoint d-cliques in $G[V_0]$. This follows from Property (**IV**)(*c*), since $t \leq \frac{1}{2d} n$. Lines 6 and 9 rely on the existence of saturating matchings in the auxiliary graphs A_i. The existence of such matchings will follow from the next two lemmas. In both, we implicitly assume the hypotheses specified in Lemma 2.

Lemma 3. *For* $i = 1, \ldots, d^2 + 2$ *and for every* $Q \subset X_i$*, we have*

$$|A_i(Q)| \geq \min\{|Q|, |W_i| - \omega p^{-d}\}. \tag{18}$$

In particular, if $|W_i| \geq |X_i| + \omega p^{-d}$*, then* $|A_i(Q)| \geq |Q|$ *for all sets* $Q \subset X_i$*.*

The graphs A_i depend on the random choice of the T_j and on f_0. Therefore, strictly speaking, the conclusions of Lemma 3 should be claimed 'with probability 1'. Our second lemma will be the key to show that, with high probability, the balanced bipartite graph A_{d^2+2} has a perfect matching; it basically asserts that small sets $Y \subset W_{d^2+2} \subset V$ expand in $A_{d^2+2} = \widetilde{A}_1[X_{d^2+2} \cup W_{d^2+2}]$ (recall (10)).

Lemma 4. *The random choice of the* T_i $(1 \le i \le t)$ *and the embedding* f_0 *of the sets* S_i $(1 \le i \le t)$ *is such that, with probability* $1 - o(1)$*, for every set* $Y \subset V$ *with* $|Y| \le \delta(4p)^{-d}$*, we have*

$$|\widetilde{A}_1(Y) \cap X_{d^2+2}| \ge \frac{1}{2}\Big(\frac{p}{5}\Big)^d t\,|Y|. \tag{19}$$

The proof of Lemma 3 is at the end of this section, while, because of length restrictions, the proof of Lemma 4, which is in fact much more involved, is omitted. The following corollary of the above two lemmas completes the proof of Lemma 2.

Corollary 1. (i) *For each* $i = 1, \dots, d^2 + 1$*, the graph* A_i *has a matching saturating* X_i*.* (ii) *The graph* A_{d^2+2} *has a perfect matching with probability* $1 - o(1)$*.*

Proof. (i) Fix $1 \le i \le d^2 + 1$ and recall that

$$W_i = V \setminus \mathrm{im}(f_{i-1}) \setminus \bigcup\nolimits_{i<j\le d^2+2} R_j$$

and, by (9), that $|W_i| \ge |X_i| + \varepsilon n$. For C sufficiently large, we have $\varepsilon n \ge C^{-d+1} n = \omega p^{-d}$. Thus, $|W_i| \ge |X_i| + \omega p^{-d}$, which, by Lemma 3, implies that $|A_i(Q)| \ge |Q|$ for all $Q \subset X_i$. Consequently, by Hall's theorem, there is a matching in A_i covering X_i.

(ii) For convenience, set $h = d^2 + 2$. To prove that A_h has a perfect matching with high probability, recall that $A_h = \widetilde{A}_1[X_h \cup W_h]$ (see (10)). By Lemma 4, with high probability, for every $Y \subset W_h$ with $|Y| \le \delta(4p)^{-d}$, we have (see (19)),

$$|A_h(Y)| = |\widetilde{A}_1(Y) \cap X_h| \ge \frac{1}{2}\Big(\frac{p}{5}\Big)^d t\,|Y| \ge \delta^{-1} 4^d \omega\,|Y|, \tag{20}$$

provided C is large enough. We claim that the conditions above ensure the existence of a perfect matching in A_h. Recall that $|X_h| = |W_h| = t$. Let $Q \subset X_h$. If $|Q| \le t - \omega p^{-d}$ then Lemma 3 implies that $|A_h(Q)| \ge |Q|$. Assume then that $|Q| \ge t - \omega p^{-d} + 1$ (for simplicity, we assume that ωp^{-d} is an integer), and suppose, for the sake of contradiction, that $|A_h(Q)| \le |Q| - 1$, equivalently, that $|W_h \setminus A_h(Q)| \ge t - |Q| + 1$. If $|W_h \setminus A_h(Q)| \le \delta(4p)^{-d}$, take $Y = W_h \setminus A_h(Q)$. Otherwise, take any $Y \subset W_h \setminus A_h(Q)$ with $|Y| = \delta(4p)^{-d}$. By (20),

$$|A_h(Y)| \ge \delta^{-1} 4^d \omega\,|Y| \ge t - |Q| + 1, \tag{21}$$

where the last inequality is clear if $Y = W_h \setminus A_h(Q)$, while, otherwise, we argue, using the definition of Y and our assumption on $|Q|$, that $\delta^{-1} 4^d \omega\,|Y| = \omega p^{-d} \ge t - |Q| + 1$. Inequality (21) contradicts the fact that $A_h(Y) \cap Q = \emptyset$. Therefore, $|A_h(Q)| \ge |Q|$ for all $Q \subset X_h$ and Hall's theorem guarantees the existence of a perfect matching in A_h. □

We close this section with the proof of Lemma 3.

Proof (of Lemma 3). Fix i with $1 \le i \le d^2 + 2$. Since X_i is 2-independent, the neighborhoods $H(x)$ are disjoint for all $x \in X_i$. For every x, we find a d-element set $D_x \subset V$ such that $f_{i-1}(H(x)) \subset D_x$ with all the D_x pairwise disjoint. Define a subgraph $A_i^* \subset A_i$ by replacing $f_{i-1}(H(x))$ with D_x in (8), that is

$$A_i^* = \big\{(x, v) \in X_i \times W_i \colon D_x \subset G(v)\big\}. \tag{22}$$

Clearly, for every $Q \subset X_i$ we have $|A_i(Q)| \ge |A_i^*(Q)|$, and, hence, it suffices to prove (18) for A_i^*. For ease of notation, we shall write A_i instead of A_i^*.

The proof is split into two cases according to whether Q is small ($|Q| \le \omega p^{-d}$) or large ($|Q| > \omega p^{-d}$). First consider the case when Q is small, and let $Q' \subset Q$ be an arbitrary subset with

$$|Q'| = \min\big\{\delta p^{-d}, |Q|\big\} \ge \frac{\delta |Q|}{\omega}. \tag{23}$$

Notice that

$$|A_i(Q')| \ge |A_i(Q') \cap R_i| = \big|\big\{w \in R_i \colon G(w) \supset D_x \text{ for some } x \in Q'\big\}\big|. \tag{24}$$

Recalling that $|Q'| \le \delta p^{-d}$ (see (23)), we apply Property (**III**)(a) to $\mathcal{X} = \{D_x \colon x \in Q'\}$ and $U = R_i$, to obtain that the cardinality of the last set in (24) is at least $(1 - 2\delta) p^d |R_i||Q'|$. In particular, for C large enough, we have

$$|A_i(Q)| \ge |A_i(Q')| \overset{(4)}{\ge} (1 - 2\delta)\varepsilon p^d n\, |Q'| \ge \delta^{-1}\omega\, |Q'| \ge |Q|.$$

Consequently, (18) holds when Q is small.

When Q is large, that is, $|Q| > \omega p^{-d}$, set $U = W_i \setminus A_i(Q)$ and suppose that $|U| \ge \omega p^{-d}$. Then, by Property (**III**)(b), there is an edge in A_i between Q and U, which is a contradiction. Thus $|U| < \omega p^{-d}$, which establishes (18). □

References

1. Alon, N., Capalbo, M.: Sparse universal graphs for bounded-degree graphs. Random Structures Algorithms 31(2), 123–133 (2007)
2. Alon, N., Capalbo, M.: Optimal universal graphs with deterministic embedding. In: Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 373–378. ACM, New York (2008)
3. Alon, N., Capalbo, M., Kohayakawa, Y., Rödl, V., Ruciński, A., Szemerédi, E.: Universality and tolerance (extended abstract). In: 41st Annual Symposium on Foundations of Computer Science, Redondo Beach, CA, pp. 14–21. IEEE Comput. Soc. Press, Los Alamitos (2000)
4. Alon, N., Capalbo, M., Kohayakawa, Y., Rödl, V., Ruciński, A., Szemerédi, E.: Near-optimum Universal Graphs for Graphs with Bounded Degrees (Extended Abstract). In: Goemans, M.X., Jansen, K., Rolim, J.D.P., Trevisan, L. (eds.) RANDOM 2001 and APPROX 2001. LNCS, vol. 2129, pp. 170–180. Springer, Heidelberg (2001)

5. Alon, N., Krivelevich, M., Sudakov, B.: Embedding nearly-spanning bounded degree trees. Combinatorica 27(6), 629–644 (2007)
6. Balogh, J., Csaba, B., Pei, M., Samotij, W.: Large bounded degree trees in expanding graphs. Electron. J. Combin. 17(1), Research Paper 6, 9 (2010)
7. Bhatt, S.N., Chung, F.R.K., Leighton, F.T., Rosenberg, A.L.: Universal graphs for bounded-degree trees and planar graphs. SIAM J. Discrete Math. 2(2), 145–155 (1989)
8. Capalbo, M.R., Kosaraju, S.R.: Small universal graphs. In: Annual ACM Symposium on Theory of Computing, Atlanta, GA, pp. 741–749 (electronic). ACM, New York (1999)
9. Dellamonica Jr., D., Kohayakawa, Y., Rödl, V., Ruciński, A.: Universality of random graphs. SIAM J. Discrete Math. (to appear)
10. Dellamonica Jr., D., Kohayakawa, Y.: An algorithmic Friedman–Pippenger theorem on tree embeddings and applications. Electron. J. Combin. 15(1), Research Paper 127, 14 (2008)
11. Dellamonica Jr., D., Kohayakawa, Y., Rödl, V., Ruciński, A.: Universality of random graphs. In: Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 782–788. ACM, New York (2008)
12. Hajnal, A., Szemerédi, E.: Proof of a conjecture of P. Erdős. In: Combinatorial Theory and its Applications II (Proc. Colloq., Balatonfüred, 1969), pp. 601–623. North-Holland, Amsterdam (1970)
13. Janson, S.: Poisson approximation for large deviations. Random Structures Algorithms 1(2), 221–229 (1990)
14. Janson, S., Łuczak, T., Rucinski, A.: Random graphs. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York (2000)
15. Janson, S., Oleszkiewicz, K., Ruciński, A.: Upper tails for subgraph counts in random graphs. Israel J. Math. 142, 61–92 (2004)
16. Johansson, A., Kahn, J., Vu, V.H.: Factors in random graphs. Random Struct. Algorithms 33(1), 1–28 (2008)
17. Kierstead, H.A., Kostochka, A.V.: A short proof of the hajnal-szemerédi theorem on equitable colouring. Combinatorics, Probability & Computing 17(2), 265–270 (2008)
18. Kierstead, H.A., Kostochka, A.V., Mydlarz, M., Szemerédi, E.: A fast algorithm for equitable coloring. Combinatorica 30(2), 217–224 (2010)