Discrete Mathematics xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

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Short paths in ε-regular pairs and small diameter decompositions of dense graphs

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ARTICLE INFO

Article history:
Received 2 October 2006
Accepted 5 November 2008
Available online xxxx

Keywords:
ε-regular graphs
Diameter
Edge decomposition

ABSTRACT

In this paper we give optimal vertex degree conditions that guarantee connection by short paths in ε-regular bipartite graphs. We also study a related question of decomposing an arbitrary graph into subgraphs of small diameter.

1. Introduction

Since its discovery in the late 1970s, the Szemerédi Regularity Lemma has been a powerful tool in modern graph theory. Roughly speaking, this deep result from [1] provides a decomposition of every large graph into bipartite subgraphs most of which are ε-regular pairs. Thus, naturally, there has been a growing interest in studying the properties of such graphs (see, e.g., [2–9]).

The goal of this paper is two-fold. First, in Section 3 we broaden our knowledge on ε-regular pairs by determining the minimum degree of vertices that guarantees small diameter. It is quite easy to show (cf. Observation 3.1) that in an ε-regular pair of order $2n$ and density $d \geq \varepsilon$ every two vertices of degree at least εn are connected by a short path. However, our main result in this direction (Theorem 3.5) shows that this bound can be replaced by a quantity of order $(\varepsilon^2/d)n$, which is essentially optimal.

The other result proved in this paper deals with graph decomposition into subgraphs of small diameter. Such decompositions have a potential application in distributed computing (see, e.g., [10]). Here we approximate a given graph by a union of subgraphs with small diameter. It is easy to prove (cf. Proposition 4.1), that for every $\gamma > 0$, the set of all but at most γn^2 edges of every n-vertex graph G can be split into no more than $1/\gamma$ subgraphs with diameter bounded from above by $3/\gamma$. In Theorem 4.5, using the Szemerédi Regularity Lemma, we push the diameter down to four at the cost of an increase in the number of parts of the decomposition.

Similar results for quasi-random 3-uniform hypergraphs have been proved in ~~[11].~~

2. Preliminaries

Let $G = (V, E)$ be a graph. For $E_0 \subseteq E$ we denote by $G[E_0] = (V_0, E_0)$, where $V_0 = \bigcup_{e \in E_0} e$, the subgraph of G induced by E_0. Note that $G[E_0]$ does not need to be an (vertex) induced subgraph in the usual sense.

By $\mathrm{dist}_G(x, y)$ we denote the *distance* of vertices $x, y \in V$, that is, the length of a shortest path connecting them, if such a path exists. Otherwise we set $\mathrm{dist}_G(x, y) = \infty$. By the *diameter* of G we mean $\mathrm{diam}(G) = \max_{x,y \in V} \mathrm{dist}_G(x, y)$. In particular, if G is not connected, then $\mathrm{diam}(G) = \infty$.

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doi:10.1016/j.disc.2008.11.023

For $A \subseteq V$, let $\mathrm{N}(A)$ be the set of all vertices adjacent in G to at least one vertex in A. In particular, if $A = \{v\}$ we write $\mathrm{N}(v)$. The number $\deg(v) = |\mathrm{N}(v)|$ is the degree of vertex v in graph G. The minimum vertex degree is denoted by δ_G. In Section 4 we will use the inequality

$$\mathrm{diam}(G) < \frac{3|V|}{\delta_G}, \tag{1}$$

valid for all connected graphs G. To verify (1), note that if $P = v_0 v_1 \cdots v_k$ is the shortest path between two vertices of distance $\mathrm{diam}(G) = k$ then the vertices $v_0, v_3, \ldots$ have disjoint neighborhoods.

For two disjoint, nonempty subsets U, W of V, we define

$$\mathrm{e}_G(U, W) = |\{\{u, w\} : u \in U, w \in W, \{u, w\} \in E\}|$$

and

$$\mathrm{d}_G(U, W) = \frac{\mathrm{e}_G(U, W)}{|U||W|}.$$

The number $\mathrm{d}_G(U, W)$ is called the *density* of the graph G between U and W, or simply, the density of the pair (U, W).

In the remainder of this section let G denote a bipartite graph with bipartition $V = V_1 \cup V_2$. A simple averaging argument yields the following fact.

Fact 2.1. *If $\mathrm{d}_G(V_1, V_2) < d$ (resp., $\mathrm{d}_G(V_1, V_2) > d$), then for all natural numbers $\ell_1 \le |V_1|$ and $\ell_2 \le |V_2|$ there exist subsets $U \subset V_1$, $|U| = \ell_1$ and $W \subset V_2$, $|W| = \ell_2$ with $\mathrm{d}_G(U, W) < d$ (resp., $\mathrm{d}_G(U, W) > d$).* ■

Now we define a central notion of our paper.

Definition 2.2. Given $\varepsilon > 0$, G is called *ε-regular* if there exists $d > 0$ such that for every pair of subsets $U \subseteq V_1$ and $W \subseteq V_2$, where $|U| \ge \varepsilon|V_1|$, $|W| \ge \varepsilon|V_2|$, the inequalities

$$d - \varepsilon < \mathrm{d}_G(U, W) < d + \varepsilon \tag{2}$$

hold. We will then also say that G, or the pair (V_1, V_2), is *(d, ε)-regular*.

Note that each ε-regular graph is ε'-regular for all $\varepsilon' \ge \varepsilon$.

The last definition of this section deals with pairs of subsets of vertices with no edge in between.

Definition 2.3. A pair of sets $U \subseteq V_1$ and $W \subseteq V_2$ with $\mathrm{e}_G(U, W) = 0$ is called a *hole* or a *$(|U|, |W|)$-hole* in G. A bipartite graph G is called *$\{\ell_1, \ell_2\}$-holeless* if there is neither an (ℓ_1, ℓ_2)-hole nor an (ℓ_2, ℓ_1)-hole in G.

Note that it follows from the definition of ε-regularity that if $\varepsilon \le d$ and $|V_1| = |V_2| = n$ then each (d, ε)-regular graph is $\{\lceil \varepsilon n \rceil, \lceil \varepsilon n \rceil\}$-holeless. However, it is proved in [12], that even much smaller holes are forbidden.

Lemma 2.4 *([12], Theorem 2.1). For all $0 < d < 1$ there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$ and $0 < \beta < \frac{2\varepsilon(\sqrt{\varepsilon}d - \varepsilon)}{d - \varepsilon} < \alpha$ there exists n_0 such that for all $n \ge n_0$*

(i) *every (d, ε)-regular bipartite graph G with $|V_1| = |V_2| = n$ is $\{\lfloor \alpha n \rfloor, \lfloor \alpha n \rfloor\}$-holeless.*
(ii) *there exists a (d, ε)-regular graph G_0 with $|V_1| = |V_2| = n$ containing a $(\lceil \beta n \rceil, \lceil \beta n \rceil)$-hole.*

3. Short paths between vertices of sufficiently large degrees

This section is devoted to the problem of what minimum degree of vertices in an ε-regular graph guarantees a small diameter. As a starting point recall an old result from [13] which says that almost all balanced bipartite graphs with density d have diameter three. Although an ε-regular pair resembles a random bipartite graph, in general nothing can be said about its diameter, because it may contain isolated vertices. On the other hand, the absence of $(\lceil \varepsilon n \rceil, \lceil \varepsilon n \rceil)$-holes implies immediately the following observation.

Observation 3.1. *For all $\varepsilon \le d$, in every (d, ε)-regular balanced bipartite graph of order $2n$ every two vertices of degree at least εn are connected by a path of length at most four.* ■

Below we first improve Observation 3.1 by relaxing the degree assumption to, roughly, $(2\varepsilon^{3/2}/\sqrt{d})n$ (Corollary 3.4), and then we show that degrees at least, roughly, $(\varepsilon^2/d)n$ yield paths of length at most five (Theorem 3.5). Moreover, both these results are, in a sense, sharp.

We first examine the relation between the existence of short paths and the absence of large holes in bipartite graphs. In what follows we consider only balanced bipartite graphs $G = (V_1 \cup V_2, E)$ in which $|V_1| = |V_2| = n$ for some n, but similar results can be also obtained for unbalanced graphs (see [12]).

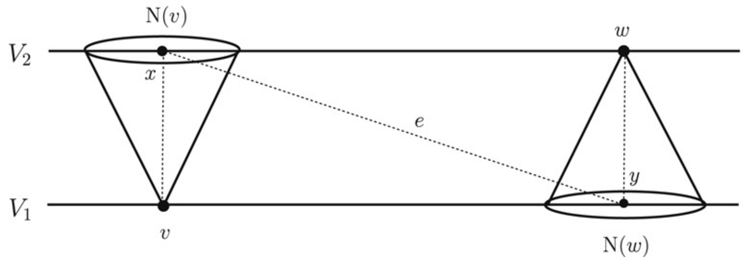

Fig. 1. A path from v to w of length three.

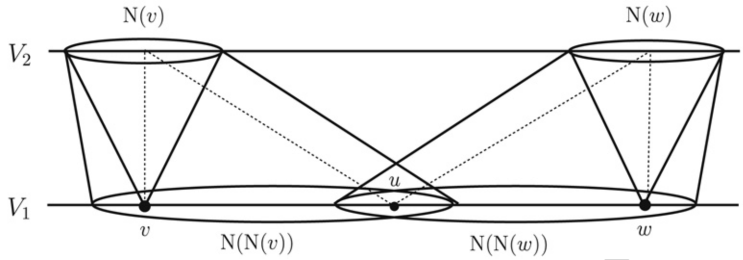

Fig. 2. A path from v to w of length four.

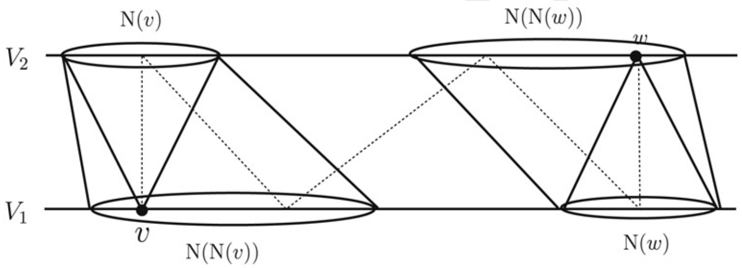

Fig. 3. The path from v to w of length five.

Proposition 3.2. *Let $\ell, \ell' \geq 1$ and G be an $\{\ell, \ell'\}$-holeless balanced bipartite graph. If $\deg(v) \geq \ell$ and $\deg(w) \geq \ell'$, then* $\mathrm{dist}_G(v, w) \leq 4$.

Proof. Without loss of generality we may assume that $v \in V_1$. Consider first the case when $w \in V_2$. By the assumption we have $|\mathrm{N}(v)| \geq \ell$ and $|\mathrm{N}(w)| \geq \ell'$, and therefore, due to the absence of (ℓ', ℓ)-holes, there exists an edge $e = \{x, y\}$ with $x \in \mathrm{N}(v)$ and $y \in \mathrm{N}(w)$. Thus, vertices v, x, y, w form in G a path from v to w of length three (see Fig. 1).

Assume now that $w \in V_1$. If $\ell + \ell' > n$ then there exists $u \in \mathrm{N}(v) \cap \mathrm{N}(w)$ and vuw is a path of length two. Otherwise, since G is $\{\ell, \ell'\}$-holeless, $|\mathrm{N}(\mathrm{N}(v))| > n - \ell'$ and similarly $|\mathrm{N}(\mathrm{N}(w))| > n - \ell$. Consequently, there exists $u \in \mathrm{N}(\mathrm{N}(v)) \cap \mathrm{N}(\mathrm{N}(w))$, and there is a path of length at most four between v and w (see Fig. 2). ■

At the cost of increasing the upper bound on the lengths of the paths by just one, we may forbid holes with just one side large, while keeping the degree threshold at the level of the smaller side of the forbidden hole.

Proposition 3.3. *Let $1 \leq \ell \leq n/2$ and G be an $\{\ell, \lceil n/2 \rceil\}$-holeless balanced bipartite graph. If $\deg(v) \geq \ell$ and $\deg(w) \geq \ell$, then* $\mathrm{dist}_G(v, w) \leq 5$.

Proof. Since G is $\{\ell, \lceil n/2 \rceil\}$-holeless,

$$\min\{|\mathrm{N}(\mathrm{N}(v))|, |\mathrm{N}(\mathrm{N}(w))|\} > n - \lceil n/2 \rceil = \lfloor n/2 \rfloor.$$

Consider first the case when $v \in V_1$ and $w \in V_2$. Since $\ell \leq \lceil n/2 \rceil$, there must be an edge between these two sets yielding a path of length at most five (see Fig. 3).

When $v, w \in V_1$, we have $\mathrm{N}(\mathrm{N}(v)) \cap \mathrm{N}(\mathrm{N}(w)) \neq \emptyset$, and consequently there is a path of length at most four between v and w (see Fig. 2). ■

Now we turn to the consequences of Propositions 3.2 and 3.3 for quasi-random graphs. In both, Corollary 3.4 and Theorem 3.5 below, we silently assume that ε is sufficiently small with respect to d and n is a sufficiently large integer.

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In other words, we suppress the assumptions stated explicitly in Lemma 2.4 above. Note that part (ii) of each of the two results asymptotically matches the bound given in part (i). Since the proofs of these results are quite similar, we only sketch how Corollary 3.4 follows from Lemma 2.4. On the other hand, we give a self-contained proof of Theorem 3.5.

Corollary 3.4. *Let* $0 < \beta < \frac{2\varepsilon(\sqrt{\varepsilon d}-\varepsilon)}{d-\varepsilon} < \alpha$.

(i) *Let G be a (d,ε)-regular balanced bipartite graph with $2n$ vertices. If*

$$\min\{\deg(v), \deg(w)\} \geq \lfloor \alpha n \rfloor$$

then $\mathrm{dist}_G(v,w) \leq 4$. *In particular, if* $\delta_G \geq \lfloor \alpha n \rfloor$ *then* $\mathrm{diam}(G) \leq 4$.

(ii) *On the other hand, there exists a (d,ε)-regular balanced bipartite graph G_0 with $2n$ vertices containing two vertices $v, w \in V$,* $\deg(v) = \deg(w) = \lceil \beta n \rceil$, *with* $\mathrm{dist}(v,w) \geq 5$.

Proof. Part (i) follows immediately from Lemma 2.4 and Proposition 3.2 (with $\ell = \ell' = \lfloor \alpha n \rfloor$). To prove Part (ii), note that, by Lemma 2.4 (ii), there exists a $(d, \varepsilon + o(1))$-regular graph with $|V_1| = |V_2| = n - 1$ containing an $(\lceil \beta n \rceil, \lceil \beta n \rceil)$-hole between sets $A \subset V_1$ and $B \subset V_2$. We add two vertices, v and w, and all the edges connecting v with the vertices of B and w with the vertices of A. This way we obtain a new graph G_0 with the desired property. ■

It turns out that based on Proposition 3.3 one can decrease the bound on vertex degrees in Corollary 3.4(i) down to, roughly, $2\varepsilon^2 n/(d+\varepsilon)$ and still get quite short paths. Again, the obtained bound is nearly optimal, namely there exist (d,ε)-regular graphs containing a vertex with degree only slightly smaller than $2\varepsilon^2 n/(d+\varepsilon)$, which is disconnected from all vertices other than its neighbors.

Theorem 3.5. *Let* $0 < \beta < \frac{2\varepsilon^2}{d+\varepsilon} < \alpha$.

(i) *Let G be a (d,ε)-regular balanced bipartite graph with $2n$ vertices. If*

$$\min\{\deg(v), \deg(w)\} \geq \lfloor \alpha n \rfloor$$

then $\mathrm{dist}_G(v,w) \leq 5$. *In particular, if* $\delta_G \geq \lfloor \alpha n \rfloor$, *then* $\mathrm{diam}(G) \leq 5$.

(ii) *On the other hand, there exists a (d,ε)-regular balanced bipartite graph G_0 with $2n$ vertices containing two vertices $v, w \in V$,* $\deg(v) = \deg(w) \geq \lceil \beta n \rceil$, *with* $\mathrm{dist}(v,w) = \infty$.

Proof. To prove the first statement, we will show that every graph G satisfying the assumptions is $\{\lfloor \alpha n \rfloor, \lceil \varepsilon n \rceil\}$-holeless, much more than we need. This, by Proposition 3.3, will imply the thesis.

Suppose, for a contradiction, that there exist in G two subsets, $A \subset V_1$ and $B \subset V_2$, with $|B| = \lceil \varepsilon n \rceil$, $|A| = \lceil 2\varepsilon |B|/(d+\varepsilon) \rceil$ and such that $\mathrm{e}_G(A,B) = 0$. Note that, for large n,

$$|V_1 \setminus A| = n - |A| \geq \varepsilon n,$$

and by the (d,ε)-regularity of G we have $\mathrm{d}_G(V_1 \setminus A, B) < d + \varepsilon$. Consequently, by Fact 2.1, one can find a subset $W \subset V_1 \setminus A$ of size $|W| = |B| - |A|$ and such that $\mathrm{d}_G(W,B) < d + \varepsilon$. Hence,

$$\mathrm{d}_G(A \cup W, B) = \frac{|W||B|\mathrm{d}_G(W,B)}{|B|^2} < \frac{(|B|-|A|)(d+\varepsilon)}{|B|} \leq d + \varepsilon - 2\varepsilon = d - \varepsilon,$$

a contradiction with the (d,ε)-regularity of G.

To prove the second statement of Theorem 3.5, we will construct the graph G_0 in two steps. First, we set

$$\gamma = \frac{2\varepsilon^2}{d+\varepsilon} - \beta$$

and note that $0 < \gamma \leq \varepsilon - \beta$. Let $G' = (V_1' \cup V_2', E)$ be a $(d+\varepsilon-\gamma, \gamma/2)$-regular graph, where $|V_1'| = n - \lceil \beta n \rceil$, $|V_2'| = n - 1$ and n is sufficiently large. The existence of such graphs can be proved easily using random graphs (see [12]).

Then, we add to V_1' a set A of $\lceil \beta n \rceil$ vertices and a vertex w to V_2', adjacent to all vertices of A, obtaining from G' a new graph G_0. We claim that the graph $G_0 = (V_1 \cup V_2, E)$, where $V_1 = V_1' \cup A$ and $V_2 = V_2' \cup \{w\}$, has the desired property. Since $G_0[A \cup \{w\}]$ is an isolated star in G_0, we have $\mathrm{dist}(v,w) = \infty$ for all $v \in V \setminus (A \cup \{w\})$. Note that $\deg(w) = |A| = \lceil \beta n \rceil$, while, provided G_0 is (d,ε)-regular, the degrees of most vertices v of G_0 are at least $(d-\varepsilon)n > \lceil \beta n \rceil$.

It remains to show that G_0 is indeed (d,ε)-regular. Let $U \subset V_1$, $W \subset V_2$, $|U|, |W| \geq \varepsilon n$. Set $U' = U \setminus A$ and $W' = W \setminus \{w\}$ and observe that $|U'| \geq \varepsilon n - \lceil \beta n \rceil > (\gamma/2)|V_1'|$, and, for large n, $|W'| > (\gamma/2)|V_2'|$ (see Fig. 4). By the $(d+\varepsilon-\gamma, \gamma/2)$-regularity of G', for sufficiently large n, we have

$$\begin{aligned}\mathrm{d}_{G_0}(U,W) &\leq \frac{\mathrm{e}_{G'}(U',W') + |A|}{|U||W|} \leq \frac{\mathrm{e}_{G'}(U',W') + |A|}{|U'||W'|} \\ &= \mathrm{d}_{G'}(U',W') + O(n^{-1}) < d + \varepsilon - \frac{1}{2}\gamma + O(n^{-1}) < d + \varepsilon,\end{aligned}$$

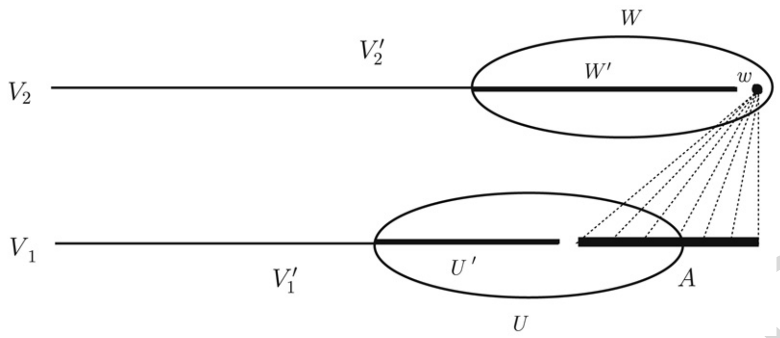

Fig. 4. Illustration of the proof of Theorem 3.5, part two.

and

$$\begin{aligned} \mathrm{d}_{G_0}(U,W) &\ge \mathrm{d}_{G'}(U',W')\left(1-\frac{|A|}{|U|}\right)\left(1-\frac{1}{|W|}\right) \\ &> \left(d+\varepsilon-\frac{3}{2}\gamma\right)\left(1-\frac{\beta}{\varepsilon}\right)-O(n^{-1}) > (d+\varepsilon-2\gamma)\left(1-\frac{2\varepsilon}{d+\varepsilon}+\frac{\gamma}{\varepsilon}\right) \\ &= d-\varepsilon+\gamma\frac{(d+\varepsilon)}{\varepsilon}-\gamma\left(2-\frac{4\varepsilon}{d+\varepsilon}+\frac{2\gamma}{\varepsilon}\right) > d-\varepsilon, \end{aligned}$$

because

$$2-\frac{4\varepsilon}{d+\varepsilon}+\frac{2\gamma}{\varepsilon} < 2 \le \frac{(d+\varepsilon)}{\varepsilon}. \quad \blacksquare$$

Note that the minimum degree of the graph G_0 constructed in the second part of the proof is just one. However, there exists an alternative construction of a disconnected (d,ε)-regular graph with minimum degree equal, roughly, to $\varepsilon^2 n/(2d)$.

4. An approximate decomposition into few subgraphs with small diameter

This section provides two results about approximating a given graph by a union of subgraphs with small diameter. As a consequence of Proposition 4.1 below, for a given $\gamma > 0$, we can split the set of all but at most $\gamma|V(G)|^2$ edges of G into no more than $1/\gamma$ subgraphs with diameter bounded from above by $3/\gamma$. Then in Theorem 4.5, using the celebrated Szemerédi Regularity Lemma, we decrease the bound on the diameter to four, at the price of increasing the number of subgraphs in the partition. Thus, in a sense, every graph can be decomposed into a bounded number of subgraphs with small diameter, provided a small set of edges and/or vertices can be ignored. The next result, which, in fact, yields a partition into vertex-disjoint subgraphs, follows easily by removing sequentially the vertices of small degrees and applying (1) to each connected component of the obtained subgraph.

Proposition 4.1. *Let $0 < \gamma < 1$. The set of edges of every graph $G = (V, E)$ with $|V| = n$ can be partitioned into $k+1$ subsets, $E = E_0 \cup E_1 \cup \cdots \cup E_k$, where $k \le 1/\gamma$ and $|E_0| \le \gamma n^2$, in such a way that for each $1 \le s \le k$ we have* $\mathrm{diam}(G[E_s]) \le 3/\gamma$. $\blacksquare$

In Proposition 4.1 both bounds, on the diameter and on the number of subgraphs $G[E_s]$, depend linearly on $1/\gamma$, and thus grow to infinity when the precision of approximation, γ, tends to zero. However, one can compromise on one of these bounds, improving the other to the extent that it becomes independent of γ. Indeed, using the Szemerédi Regularity Lemma [1] and Corollary 3.4(ii), we will put the cap of four on the diameter, at the cost of letting the number of subgraphs in the partition be an enormous constant. Before we make this precise, let us quote the Regularity Lemma.

Definition 4.2. Let $0 < \varepsilon < 1$, t be a positive integer, and let $G = (V, E)$ be a graph. We call a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ *ε-regular*, if $|V_1| = |V_2| = \cdots = |V_t|$, $|V_0| < t$, and all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \le i < j \le t$, are ε-regular.

Theorem 4.3 (*Szemerédi Regularity Lemma, [1]*). *Let $0 < \varepsilon < 1$ and let t_0 be a positive integer. There exist integers $N = N(\varepsilon, t_0)$ and $T = T(\varepsilon, t_0)$ such that every graph $G = (V, E)$ with $|V| \ge N$ vertices admits an ε-regular partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ with $t_0 \le t \le T$.*

In the proof of Theorem 4.5, we will also need a simple observation that for $\varepsilon \le d/3$, every (d,ε)-regular graph can be approximated by a subgraph of diameter at most four. It can be easily proved by removing the vertices of degrees at most $2\varepsilon n$ and applying Observation 3.1 to the resulting subgraph.

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Fact 4.4. *If $0 < 3\varepsilon \le d < 1$ then every (d, ε)-regular balanced bipartite graph G with $2n$ vertices contains an induced subgraph G' such that $|E(G) \setminus E(G')| \le 2\varepsilon n^2$ and $\operatorname{diam}(G') \le 4$.* ■

Now we are ready to prove the main result of this section.

Theorem 4.5. *Let $0 < \gamma < 1$. There exist positive integers K and N such that the set of edges of every graph $G = (V, E)$ with $|V| = n \ge N$ can be partitioned into $k + 1$ subsets, $E = E_0 \cup E_1 \cup \ldots \cup E_k$, where $k \le K$ and $|E_0| \le \gamma n^2$, in such a way that for each $1 \le s \le k$ we have $\operatorname{diam}(G[E_s]) \le 4$.*

Proof. Let $\varepsilon = \gamma/5$ and $t_0 = \lceil 1/\varepsilon \rceil$, and let $N' = N'(\varepsilon, t_0)$ and $T = T(\varepsilon, t_0)$ be given by the Szemerédi Regularity Lemma (Theorem 4.3).

Set $K = \binom{T}{2}$ and $N = \max\{T/\varepsilon, N'\}$ and let $G = (V, E)$ be a graph with $|V| = n \ge N$. Apply Theorem 4.3 to G obtaining an ε-regular partition $V = V_0 \cup V_1 \cup V_2 \cup \cdots \cup V_t$, where $t_0 \le t \le T$.

Let F_1 be the set of all edges of G intersecting V_0. Then

$$|F_1| < tn \le Tn \le \varepsilon N n \le \frac{\gamma}{5} n^2.$$

Let F_2 be the set of all edges of G contained in one of the sets V_i, $i = 1, 2, \ldots, t$. We have

$$|F_2| \le t \binom{\lfloor n/t \rfloor}{2} < \frac{n^2}{2t} \le \frac{n^2}{2t_0} \le \frac{\gamma}{10} n^2.$$

Further, let F_3 be the set of all edges belonging to the pairs (V_i, V_j) which are not ε-regular. By the ε-regularity of the partition,

$$|F_3| \le \varepsilon \binom{t}{2} \left(\frac{n}{t}\right)^2 < \frac{\varepsilon}{2} n^2 = \frac{\gamma}{10} n^2.$$

Finally, let F_4 be the set of all edges belonging to the pairs (V_i, V_j) with density $d_G(V_i, V_j) \le 4\varepsilon$. Then

$$|F_4| \le \binom{t}{2} 4\varepsilon \left(\frac{n}{t}\right)^2 < 2\varepsilon n^2 = \frac{2\gamma}{5} n^2.$$

Consequently, setting $E_0' = F_1 \cup F_2 \cup F_3 \cup F_4$, we have

$$|E_0'| < \frac{\gamma}{5} n^2 + \frac{\gamma}{10} n^2 + \frac{\gamma}{10} n^2 + \frac{2\gamma}{5} n^2 = \frac{4}{5} \gamma n^2.$$

Denote by E_s', $1 \le s \le \binom{t}{2}$, the sets of edges $E_G(V_i, V_j) \setminus E_0'$ which are nonempty. Without loss of generality we may assume that these are the sets $E_1', \ldots, E_k'$, where $k \le \binom{t}{2} \le K$. Note that the subgraphs $G[E_s']$ are (d, ε)-regular for some $d \ge 3\varepsilon$, and thus, according to Fact 4.4, one can remove from each of them a set E_s^0 of edges,

$$|E_s^0| \le 2\varepsilon \left(\frac{n}{t}\right)^2 = \frac{2}{5} \gamma \left(\frac{n}{t}\right)^2,$$

such that $\operatorname{diam}(G[E_s' \setminus E_s^0]) \le 4$. Altogether we have deleted a set $E_0'' = \bigcup_{s=1}^k E_s^0$ of at most

$$\binom{t}{2} \frac{2}{5} \gamma \left(\frac{n}{t}\right)^2 < \frac{1}{5} \gamma n^2$$

edges. Since

$$|E_0' \cup E_0''| < \frac{4}{5} \gamma n^2 + \frac{1}{5} \gamma n^2 = \gamma n^2,$$

we obtain the desired partition by setting $E_0 = E_0' \cup E_0''$, and $E_s = E_s' \setminus E_s^0$, $1 \le s \le k \le K$. ■

Note that the proofs of Proposition 4.1 and Theorem 4.5 both yield algorithms of complexity $O(n^2)$ which construct the respective partitions. The latter is based on an $O(n^2)$-time algorithm found in [14] which builds an ε-regular partition in every n-vertex graph.

Uncited references

[15] and [16].

Acknowledgements

We would like to thank both referees for their valuable comments on the presentation of our paper. We are also most grateful to Brendan Nagle who suggested looking at the problem of minimum degree guaranteeing paths of length at most four (Corollary 3.4) and sketched a solution to it. He also supplied us with an alternative construction for the proof of Theorem 3.5(ii) and observed the algorithmic aspects of Proposition 4.1 and Theorem 4.5 discussed in Section 4.

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Please cite this article in press as: J. Polcyn, A. Ruciński, Short paths in ε-regular pairs and small diameter decompositions of dense graphs, Discrete Mathematics (2008), doi:10.1016/j.disc.2008.11.023