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On the Number of Perfect Matchings in Random Lifts

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Let G be a fixed connected multigraph with no loops. A random n-lift of G is obtained by replacing each vertex of G by a set of n vertices (where these sets are pairwise disjoint) and replacing each edge by a randomly chosen perfect matching between the n-sets corresponding to the endpoints of the edge. Let X_G be the number of perfect matchings in a random lift of G. We study the distribution of X_G in the limit as n tends to infinity, using the small subgraph conditioning method.

We present several results including an asymptotic formula for the expectation of X_G when G is d-regular, $d \geqslant 3$. The interaction of perfect matchings with short cycles in random lifts of regular multigraphs is also analysed. Partial calculations are performed for the second moment of X_G, with full details given for two example multigraphs, including the complete graph K_4.

To assist in our calculations we provide a theorem for estimating a summation over multiple dimensions using Laplace's method. This result is phrased as a summation over lattice points, and may prove useful in future applications.

1. Introduction

Throughout, let G be a fixed connected multigraph with g vertices and no loops. For simplicity we assume that $V(G) = [g] := \{1, \ldots, g\}$. A random n-lift of G is a random graph on the vertex set $V_1 \cup V_2 \cup \cdots \cup V_g$, where each V_i is a set of n vertices and these sets are pairwise disjoint, obtained by placing a uniformly chosen random perfect matching between V_i and V_j, independently for each edge $e = ij$ of G. Denote the resulting random

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graph by $L_n(G)$. The perfect matching corresponding to the edge e of G is called the *fibre* corresponding to e, which we denote by F_e. Note that the degree of $v \in V_i$ in $L_n(G)$ is equal to the degree $d_G(i)$ of vertex i in G. In particular, if G is d-regular, then so is $L_n(G)$. We are interested in asymptotics as n tends to infinity.

This model of sparse random graphs was introduced and studied in a series of papers by Amit, Linial, Matoušek and Rozenman [2, 3, 4, 12]. Linial and Rozenman [12] studied the existence of a perfect matching in $L_n(G)$ and described a large class of graphs G for which $L_n(G)$ a.a.s. contains a perfect matching (for n even, at least). This class contains all regular graphs and, in turn, is contained in the class of graphs having a fractional perfect matching (see Section 3 for a definition). Observe that if G has a perfect matching then every lift of G has at least one perfect matching.

In this paper we study the number of perfect matchings in $L_n(G)$ in the limit as n tends to infinity, where G is a graph with a fractional perfect matching. To do this we use the *small subgraph conditioning method*, which provides a concentration result based on the second moment method conditioned on the number of small cycles. For a concise description of the method, see [11, Theorems 9.12 and 9.13].

Let X_G be the number of perfect matchings in $L_n(G)$. To apply the small subgraph conditioning method, asymptotic expressions for $\mathbb{E}\, X_G$ and $\mathbb{E}(X_G^2)$ must be found. Then the limit of the ratio $\mathbb{E}(X_G^2)/(\mathbb{E}\, X_G)^2$ is compared with a quantity which depends upon the interaction of perfect matchings and short cycles in $L_n(G)$.

In Sections 3 and 4 we write the first and second moments of X_G as multiple sums of some explicit terms, and then estimate the sums by Laplace's method. This is a standard method for similar moment estimates, and in particular, it has been used in several papers on random regular graphs. (See, for example, [11, Chapter 9] and the references given there.) However, in the present paper, each summation is over an index set of rather high dimension with a number of side conditions on the indices, while in many previous applications the summations are only over one or two variables. To assist with these calculations, we present a general theorem (Theorem 2.3) that encapsulates Laplace's method for a general situation, with sums over a lattice in a subspace of $\mathbb{R}^N$. We do this both because we think that it clarifies the argument in the present work, and because we hope that it might be useful in future applications. The necessary terminology and notation is introduced in Section 2, where Theorem 2.3 is stated. The proof of Theorem 2.3 can be found in Section 6.

Using this machinery we prove an asymptotic formula for $\mathbb{E}\, X_G$ for any connected regular multigraph G with degree at least three (see Theorem 3.3). However, two difficulties (one algebraic and one analytic) have prevented us from obtaining an asymptotic formula for $\mathbb{E}(X_G^2)$ in the same generality, though we have partial results in Theorem 4.2 and Lemma 4.3. We illustrate these results by calculating $\mathbb{E}(X_G^2)$ for two multigraphs: specifically, for the complete graph K_4 and for the multigraph consisting of two vertices and three parallel edges, which we denote by K_2^3. These calculations were performed with the aid of `Maple`. (A file containing the `Maple` commands is available from [20].)

In Section 5 we prove the necessary results relating to short cycles in random lifts (Lemmas 5.1, 5.2 and Corollary 5.4). As corollaries, using [11, Theorem 9.12] we obtain a concentration result for X_G in our two illustrative examples (see Corollaries 5.5 and 5.6).

One of the most interesting questions on random lifts is the problem of existence of a Hamilton cycle. There is a conjecture (attributed to Linial) that a random lift of K_4 is a.a.s. Hamiltonian. Indeed, we believe that a.a.s. $L_n(G)$ is Hamiltonian for all connected d-regular loop-free multigraphs G with $d \geqslant 3$. (This is known to be true when G is a multigraph with exactly two vertices and at least three edges: see Remark 1 below.) Burgin, Chebolu, Cooper and Frieze [6] showed that a.a.s. $L_n(K_g)$ is Hamiltonian when g is large enough (see also [7] for the directed case). The arguments in [6] are combinatorial and utilize the celebrated idea of Pósa. For small g, we feel that the small subgraph conditioning method may be a fruitful line of attack, as it has been very successful for studying Hamilton cycles in random regular graphs (Robinson and Wormald [17, 18]; see also [11, Chapter 9]). This remains an open problem.

Remark 1. We allow the multigraph G to have multiple edges. The simplest case is when G consists of only two vertices, with d parallel edges between them. The random lift $L_n(G)$ is then a random bipartite (multi)graph obtained by taking the union of d independent random matchings between two sets of n vertices each. Such sums have been studied in [15], where they were shown to be contiguous to random bipartite d-regular (multi)graphs. The latter, in turn, is known to be a.a.s. Hamiltonian (see [16] for a standard, second moment method proof). Hence, for this small multigraph G with $d \geqslant 3$, the random lift $L_n(G)$ is a.a.s. Hamiltonian too.

Remark 2. Random lifts of multigraphs with loops can also be formed. As in [2], the fibre corresponding to a loop is given by the n edges $i\sigma(i)$ for a random permutation σ of $[n]$. This is a random 2-regular (multi)graph, denoted by $\mathbb{P}(n)$ in [11, Remark 9.45]. While we do not allow loops in our current work, for several reasons, we believe that the results here can be extended to multigraphs with loops. A simple and interesting case is when G consists of a single vertex with $d/2$ loops (d even). Then $L_n(G)$ consists of the sum (union) of $d/2$ independent copies of $\mathbb{P}(n)$. Such sums have been shown to be contiguous to random d-regular (multi)graphs in [8].

2. Notation, terminology and a summation theorem

As mentioned above, G denotes a fixed connected multigraph with g vertices and no loops. For simplicity we assume that $V(G) = [g] := \{1, \dots, g\}$. We denote the number of edges in G by h. (Often we assume G to be d-regular, and then $h = dg/2$.) Let $A = A_G$ be the $g \times g$ adjacency matrix of G and let $\widehat{A} = \widehat{A}_G$ be the incidence matrix of G, with g rows and h columns. Thus

$$\widehat{A}\widehat{A}^T = A + D_G, \tag{2.1}$$

where D_G is the diagonal matrix with entries $d_G(i)$, $i \in V(G)$. Denote the eigenvalues of A by $\alpha_1, \dots, \alpha_g$.

In Section 4 we also need a directed incidence matrix for G. Give each edge in G an (arbitrary) direction, and let $\vec{A}_G$ be the corresponding directed incidence matrix. In other words, $\vec{A}_G$ is the $g \times h$ matrix obtained from $\widehat{A}$ by changing the sign of one of the two 1s in each column. Then

$$\vec{A}_G \vec{A}_G^T = D_G - A. \tag{2.2}$$

Our version of Laplace's method (Theorem 2.3) involves lattices. A *lattice* is a discrete subgroup of $\mathbb{R}^N$. (Discrete means that the intersection with any bounded set in $\mathbb{R}^N$ is finite.) It is well known that every lattice $\mathcal{L}$ is isomorphic (as a group) to $\mathbb{Z}^r$ for some r with $0 \leqslant r \leqslant n$. The integer r is called the *rank* of $\mathcal{L}$ and is denoted by $\operatorname{rank}(\mathcal{L})$. In other words, every lattice $\mathcal{L}$ has a *basis*, *i.e.*, a sequence $x_1, \dots, x_r$ of elements of $\mathcal{L}$ such that every element of $\mathcal{L}$ has a unique representation $\sum_{i=1}^r n_i x_i$ with $n_i \in \mathbb{Z}$. Furthermore, the basis elements $x_1, \dots, x_r$ are linearly independent (over $\mathbb{R}$); thus the rank equals the dimension of the linear subspace spanned by $\mathcal{L}$.

The basis is not unique (except in the trivial case $r = 0$); if $\Xi = (\xi_{ij})$ is any $r \times r$ integer matrix such that the determinant $\det(\Xi) = \pm 1$ (which is equivalent to the condition that both Ξ and Ξ^{-1} are integer matrices) and $(x_i)_1^r$ is a basis of $\mathcal{L}$, then $y_i = \sum_j \xi_{ij} x_j$ defines another basis $y_1, \dots, y_r$; conversely, given $(x_i)_1^r$, every basis of $\mathcal{L}$ is obtained in this way by some such matrix Ξ.

A *unit cell* of the lattice $\mathcal{L}$ is the set $\{\sum_{i=1}^r t_i x_i : 0 \leqslant t_i < 1\}$ for some basis $(x_i)_i$ of $\mathcal{L}$. If $\mathcal{L} \subset \mathbb{R}^N$ has full rank N, and U is any unit cell of $\mathcal{L}$, then $\{x + U\}_{x \in \mathcal{L}}$ is a partition of $\mathbb{R}^N$.

The unit cells of a lattice $\mathcal{L}$ all have the same r-dimensional volume (Hausdorff measure), where $r = \operatorname{rank}(\mathcal{L})$; this volume is the *determinant* (or *covolume*) of $\mathcal{L}$, and is denoted by $\det(\mathcal{L})$.

If $(x_i)_{i=1}^r$ is a sequence of vectors in $\mathbb{R}^N$, the symmetric matrix $(\langle x_i, x_j \rangle)_{i,j=1}^r$ of their inner products is called their *Gram matrix*. It is well known that $x_1, \dots, x_r$ are linearly independent if and only if the Gram matrix is non-singular, *i.e.*, if and only if the *Gram determinant* $\det(\langle x_i, x_j \rangle)_{i,j=1}^r \neq 0$.

The following results are well known.

Lemma 2.1. *If $(x_i)_{i=1}^r$ is a basis of a lattice $\mathcal{L}$ in $\mathbb{R}^N$, then*

$$\det(\langle x_i, x_j \rangle)_{i,j=1}^r = \det(\mathcal{L})^2. \tag{2.3}$$

Lemma 2.2. *If $\mathcal{L}_1 \subseteq \mathcal{L}_2$ are two lattices of the same rank, then $\mathcal{L}_2/\mathcal{L}_1$ is a finite group of order* $\det(\mathcal{L}_1)/\det(\mathcal{L}_2)$.

The *Hessian* or second derivative $D^2\phi(x_0)$ of a function ϕ at a point $x_0 \in \mathbb{R}^N$ is an $N \times N$ matrix; it is also naturally regarded as a bilinear form on $\mathbb{R}^N$. In general, if B is a bilinear form on $\mathbb{R}^N$, it corresponds to the matrix $(B(e_i, e_j))_{i,j=1}^N$, where $(e_i)_{i=1}^N$ is the standard basis. We define the determinant $\det(B)$ as $\det(B(e_i, e_j))_{i,j=1}^N$, and note that if $z_1, \dots, z_N$ is any basis in $\mathbb{R}^N$, then

$$\det(B) = \frac{\det\big(B(z_i, z_j)\big)_{i,j=1}^N}{\det(\langle z_i, z_j \rangle)_{i,j=1}^N}. \tag{2.4}$$

We are interested in the restriction to a subspace. If B is a bilinear form on $\mathbb{R}^N$ and $V \subseteq \mathbb{R}^N$ is a subspace, we let $\det(B|_V)$ denote the determinant of B regarded as a bilinear form on V. By (2.4), this can be computed as

$$\det(B|_V) = \frac{\det\bigl(B(z_i, z_j)\bigr)_{i,j=1}^r}{\det(\langle z_i, z_j\rangle)_{i,j=1}^r}. \tag{2.5}$$

for any basis $z_1, \dots, z_r$ of V.

We now state our general theorem for performing summation over a lattice using Laplace's method.

Theorem 2.3. *Suppose the following.*

(i) $\mathcal{L} \subset \mathbb{R}^N$ *is a lattice with rank* $r \leqslant N$.

(ii) $V \subseteq \mathbb{R}^N$ *is the r-dimensional subspace spanned by* $\mathcal{L}$.

(iii) $W = V + w$ *is an affine subspace parallel to* V*, for some* $w \in \mathbb{R}^N$.

(iv) $K \subset \mathbb{R}^N$ *is a compact convex set with non-empty interior* K°.

(v) $\phi : K \to \mathbb{R}$ *is a continuous function and the restriction of* ϕ *to* $K \cap W$ *has a unique maximum at some point* $x_0 \in K^\circ \cap W$.

(vi) ϕ *is twice continuously differentiable in a neighbourhood of* x_0 *and* $H := D^2\phi(x_0)$ *is its Hessian at* x_0.

(vii) $\psi : K_1 \to \mathbb{R}$ *is a continuous function on some neighbourhood* $K_1 \subseteq K$ *of* x_0 *with* $\psi(x_0) > 0$.

(viii) *For each positive integer n there is a vector* $\ell_n \in \mathbb{R}^N$ *with* $\ell_n/n \in W$.

(ix) *For each positive integer n there is a positive real number* b_n *and a function* $a_n : (\mathcal{L} + \ell_n) \cap nK \to \mathbb{R}$ *such that, as* $n \to \infty$,

$$a_n(\ell) = O\bigl(b_n e^{n\phi(\ell/n)+o(n)}\bigr), \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK, \tag{2.6}$$

and

$$a_n(\ell) = b_n(\psi(\ell/n) + o(1))e^{n\phi(\ell/n)}, \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK_1,$$

uniformly for ℓ *in the indicated sets.*

Then, provided $\det(-H|_V) \neq 0$*, as* $n \to \infty$,

$$\sum_{\ell \in (\mathcal{L}+\ell_n)\cap nK} a_n(\ell) \sim \frac{(2\pi)^{r/2}\psi(x_0)}{\det(\mathcal{L})\det(-H|_V)^{1/2}} b_n n^{r/2} e^{n\phi(x_0)}. \tag{2.7}$$

We remark that Theorem 2.3 can be generalized to allow n to tend to infinity along any infinite subset I of the positive integers, with the same proof. (Then (viii) and (ix) need only hold for every $n \in I$.)

3. Expected number of perfect matchings

A *fractional perfect matching* of the multigraph G is a function $f : E(G) \to [0, 1]$ such that

$$\sum_{e \ni v} f(e) = 1 \quad \text{for all } v \in V(G).$$

Note that every d-regular multigraph has a trivial fractional perfect matching obtained by giving each edge weight $1/d$. We often treat f as a vector $(f(e))_{e\in E(G)}$.

First, note that if there is a perfect matching at all in a lift $L_n(G)$ of G, then there exists a fractional perfect matching f of G such that $nf(e)$ is an integer for each e. Indeed, suppose that M is a perfect matching of a lift of G. Let ℓ_e be the number of edges from the fibre F_e in M, for each edge $e \in E(G)$. Then the function $f : E(G) \to [0,1]$ defined by $f(e) = \ell_e/n$ is a fractional perfect matching of G. Conversely, suppose that there exists a fractional perfect matching $z = (z_e)_e$ in G such that nz_e is an integer for each e. We may construct an n-lift of G that contains a perfect matching as follows. First take nz_e edges above each edge $e \in E(G)$, with all their endpoints disjoint. This yields n endpoints above each vertex $i \in G$, so we have constructed the sets V_i, and a perfect matching. Extend this perfect matching to an n-lift by adding further edges between V_i and V_j for all edges $e = ij$. Consequently, $L_n(G)$ has a perfect matching with positive probability if and only if there exists a fractional perfect matching z with nz integer-valued. From now on, for a given graph G we consider only those values of n for which this holds, since otherwise trivially $X_G = 0$.

Remark 3. It seems an interesting problem to characterize the set of such n for a given graph, but this is outside the scope of the present paper, and we note only the following examples. If G itself has a perfect matching then every n is allowed. On the other hand, if g is odd, then only even n are possible. If G is of odd order and Hamiltonian, then the set of allowed n is exactly the set of positive even integers. If G is d-regular, then $(1/d,\dots,1/d)$ is a fractional perfect matching, so every multiple of d is an allowed n (but there might be others too). The result by Linial and Rozenman [12] implies that for a large class of graphs defined there, every large even n is allowed. Note finally that if n_1 and n_2 are allowed, then so is $n_1 + n_2$. Hence the set of allowed n is always infinite, unless it is empty, so it makes sense to talk about asymptotic results.

Suppose that there exists a fractional perfect matching $z = (z_e)_e$ in G with nz an integer vector. If a perfect matching in $L_n(G)$ has ℓ_e edges in the fibre F_e over e, then $\sum_{e\ni v} \ell_e = n = n\sum_{e\ni v} z_e$ for every e, so $(\ell_e)_e - nz$ belongs to the lattice $\mathcal{L}_G^{(1)}$ in $\mathbb{R}^{E(G)}$ defined by

$$\begin{aligned}\mathcal{L}_G^{(1)} &:= \Big\{(v_e)_e \in \mathbb{Z}^{E(G)} : \sum_{e\ni v} v_e = 0 \quad \text{for every } v \in V(G)\Big\}\\ &= \{v \in \mathbb{Z}^{E(G)} : \widehat{A}v = 0\}.\end{aligned}$$

(The superscript 1 denotes the first moment.) Here, and elsewhere when convenient, we think of the vectors as column vectors although we write them as row vectors for typographical reasons. Conversely, if $\ell = (\ell_e)_e$ is a vector such that $\ell - nz \in \mathcal{L}_G^{(1)}$, then ℓ is an integer vector and $\sum_{e\ni v} \ell_e = \sum_{e\ni v} nz_e = n$ for every v.

Given such an integer vector $(\ell_e)_e \in \mathcal{L}_G^{(1)} + nz$, let us compute the expected number of perfect matchings in $L_n(G)$ with ℓ_e edges in the fibre F_e. Clearly this number is zero unless

$0 \leqslant \ell_e \leqslant n$ for all e. Then the endpoints of the edges in the matching may be chosen in

$$\prod_{v \in V(G)} \frac{n!}{\prod_{e \ni v} \ell_e!} = n!^g \prod_e (\ell_e!)^{-2}$$

ways, and for each choice, there are $\ell_e!(n-\ell_e)!$ possibilities for the fibre F_e, with probability $1/n!$ each. Hence, defining $K = [0,1]^{E(G)}$ we have

$$\mathbb{E}(X_G) = \sum_{\ell \in (\mathcal{L}_G^{(1)} + nz) \cap nK} a_n(\ell), \tag{3.1}$$

where

$$a_n(\ell) := n!^{g-h} \prod_e \frac{(n-\ell_e)!}{\ell_e!}.$$

(Recall that h denotes the number of edges in G.)

We wish to evaluate the sum (3.1) asymptotically by Laplace's method: more precisely, by applying Theorem 2.3. We use Stirling's formula in the following form, valid for all $n \geqslant 0$, where $x \vee y := \max(x, y)$:

$$\ln(n!) = n \ln n - n + \tfrac{1}{2}\ln(n \vee 1) + \tfrac{1}{2}\ln 2\pi + O(1/(n+1)). \tag{3.2}$$

Let $x_e = \ell_e/n$ for all $e \in E(G)$. Applying (3.2) we obtain, uniformly for $\ell \in (\mathcal{L}_G^{(1)} + nz) \cap nK$,

$$\begin{aligned}
\ln(a_n(\ell)) &= (g-h)\ln(n!) + \sum_{e \in E(G)} \big(\ln((n-\ell_e)!) - \ln(\ell_e!)\big) \\
&= (g-h)\big(n(\ln(n)-1) + \tfrac{1}{2}\ln(n) + \tfrac{1}{2}\ln(2\pi) + O(1/n)\big) \\
&\quad + \sum_{e \in E(G)} (n - 2\ell_e)(\ln(n)-1) + n \sum_{e \in E(G)} ((1-x_e)\ln(1-x_e) - x_e \ln(x_e)) \\
&\quad + \frac{1}{2}\sum_{e \in E(G)} (\ln((1-x_e) \vee n^{-1}) - \ln(x_e \vee n^{-1})) + \sum_{e \in E(G)} O\left(\frac{1}{\ell_e + 1} + \frac{1}{n - \ell_e + 1}\right).
\end{aligned}$$

Since

$$\sum_{e \in E(G)} \ell_e = \tfrac{1}{2}\sum_v \sum_{e \ni v} \ell_e = \tfrac{1}{2}\sum_v n = \tfrac{1}{2} g n,$$

after cancellation, $a_n(\ell)$ can be expressed as

$$a_n(\ell) = b_n\, \psi(\ell/n)\, \exp(n\phi(\ell/n))\left(1 + O\left(\frac{1}{\min \ell_e + 1}\right) + O\left(\frac{1}{n - \max \ell_e + 1}\right)\right)$$

where, for $x \in \mathbb{R}^{E(G)}$,

$$b_n := (2\pi n)^{(g-h)/2}, \tag{3.3}$$

$$\phi(x) := \sum_e ((1-x_e)\ln(1-x_e) - x_e \ln(x_e)), \tag{3.4}$$

$$\psi(x) := \prod_e \left(\frac{1-x_e}{x_e}\right)^{1/2}, \tag{3.5}$$

except that if some x_e or $1-x_e$ is 0, we replace it by $1/n$ in (3.5). This implies that $a_n(\ell)$ satisfies condition (2.6) of Theorem 2.3 with the above b_n, ϕ, and ψ. We will now check all the remaining assumptions of Theorem 2.3. Let

$$W := \left\{ x = (x_e) \in \mathbb{R}^{E(G)} : \sum_{e \ni v} x_e = 1 \quad \text{for every } v \in V(G) \right\} = \{x : \widehat{A}x = (1, \dots, 1)\}.$$

As is well known, and described in Section 6 in detail, the sum (3.1) is dominated by the terms where $\phi(\ell/n)$ is close to its maximum. In order to find the maximum, we restrict ourselves to regular multigraphs, where the result is simple. (The method applies to other graphs as well, provided one can find the maximum point(s) of ϕ.)

Lemma 3.1. *Suppose that G is d-regular, where $d \geqslant 3$. Then ϕ defined by* (3.4) *has a unique maximum on $K \cap W = \{x \in K : \widehat{A}x = (1, \dots, 1)\}$, attained at the point $x^0 = (1/d, \dots, 1/d)$. The maximum value is*

$$\phi(x^0) = \frac{g}{2} \ln\left(\frac{(d-1)^{d-1}}{d^{d-2}} \right),$$

and, for ψ in (3.5) *and the Hessian $D^2\phi$,*

$$\psi(x^0) = (d-1)^{h/2}, \qquad D^2\phi(x^0) = -\frac{d(d-2)}{d-1} I.$$

Proof. We write $\phi = \frac{1}{2}\sum_{v \in V(G)} \phi_v$, where

$$\phi_v(x_e : e \ni v) = \sum_{e \ni v} ((1-x_e)\ln(1-x_e) - x_e \ln(x_e)). \tag{3.6}$$

Fix a vertex $v \in V(G)$. We rename the variables x_e, $e \ni v$, by $x_1, \dots, x_d$, for convenience. Since ϕ_v is continuous, it has a maximum over the compact set

$$\Sigma_d := \left\{ (x_i)_i \in [0,1]^d : \sum_1^d x_i = 1 \right\}.$$

Let $x^v \in \Sigma_d$ be a maximum point of ϕ_v. Assume first that x^v is an interior point, *i.e.*, that $x^v \in (0,1)^d$. Then the function $f(y) = \phi_v(x_1^v + y, x_2^v - y, x_3^v, \dots, x_d^v)$ achieves a maximum at $y = 0$. Therefore, $f'(0) = 0$ and, by the chain rule,

$$\frac{\partial \phi_v(x)}{\partial x_1}(x^v) = \frac{\partial \phi_v(x)}{\partial x_2}(x^v).$$

By the same argument (or by the general Lagrange multiplier method), we have that for some constant $c_v > 0$

$$\frac{\partial \phi_v(x)}{\partial x_i}(x^v) = c_v, \quad \text{for } i = 1, \dots, d.$$

But

$$\frac{\partial \phi_v(x)}{\partial x_i}(x^v) = -\ln(1-x_i) - \ln x_i - 2,$$

so

$$x_i^v(1 - x_i^v) = \exp\{-c_v - 2\} \quad \text{for all } i = 1, \ldots, d.$$

This implies that the x_i^vs are all at the same distance from $1/2$. That is, for some constant $c_v' \geqslant 0$ we have $x_i^v = 1/2 \pm c_v'$ for $i = 1, \ldots, d$. Since $\sum_i x_i^v = 1$ and $d \geqslant 3$, we have to choose the minus sign for all i, and thus all x_i^v are equal. Since $x^v \in \Sigma_d$ we conclude that $x_i^v = 1/d$ for $i = 1, \ldots, d$.

We also have to consider the boundary of Σ_d. If, say, $x_1^v = 0$ and $0 < x_2^v < 1$, then f above is defined for small positive y with $f'(0+) = +\infty$, so x^v cannot be a maximum point on Σ_d. The only remaining points are those with all $x_i \in \{0, 1\}$, but then $\phi_v(x) = 0$, while $\phi_v(1/d, \ldots, 1/d) > 0$, so these too cannot be (global) maximum points. Hence x^v is the unique maximum point for ϕ_v on Σ_d.

Setting $x^0 = (1/d, \ldots, 1/d) \in \mathbb{R}^g$, we have for all $x \in K \cap W$

$$\phi(x) \leqslant \tfrac{1}{2} \sum_v \phi_v(x^v) = \phi(x^0).$$

Moreover, the inequality is strict for all $x \neq x^0$. This proves that x^0 is a unique maximum point of ϕ in $K \cap W$. Clearly, x^0 belongs to the interior of K. Moreover, $\phi(x^0)$ and $\psi(x^0)$ are given by the formulas stated in Lemma 3.1.

Finally, the Hessian $D^2\phi(x)$ is diagonal with entries $(1 - x_e)^{-1} - x_e^{-1}$. Hence, at x^0 we have $D^2\phi(x^0) = -\frac{d(d-2)}{d-1} I$. □

We have verified all assumptions of Theorem 2.3, for any neighbourhood K_1 of x^0 with $\overline{K_1} \subset K^\circ$. To apply formula (2.7), we still need to compute the rank of the lattice $\mathcal{L}_G^{(1)}$ and its determinant $\det(\mathcal{L}_G^{(1)})$.

Lemma 3.2.

(i) *If G is non-bipartite then the lattice $\mathcal{L}_G^{(1)}$ has rank $h - g$ and determinant* $\det(\mathcal{L}_G^{(1)}) = \frac{1}{2} \det(A + D_G)^{1/2}$.

(ii) *If G is bipartite then the lattice $\mathcal{L}_G^{(1)}$ has rank $h - g + 1$ and determinant* $\det(\mathcal{L}_G^{(1)}) = \det(A' + D_G')^{1/2}$, *where the matrix A' (respectively, D_G') is obtained by deleting the last row and column of A (respectively, D_G).*

Proof. For $v \in V(G)$ define the vector $x^v = \big(\mathbf{1}[v \in e], e \in E(G)\big)$ given by the row of the incidence matrix $\widehat{A}$ corresponding to v. For convenience, rename these vectors $x_1, \ldots, x_g$. Then, by (2.1), the Gram matrix of $x_1, \ldots, x_g$ is $\widehat{A}\widehat{A}^T = A + D_G$. This matrix is singular if and only if there exists a non-zero vector $y = (y_v) \in \mathbb{R}^{V(G)}$ with $y\widehat{A} = 0$. This is equivalent to $y_i = -y_j$ for every edge ij, and it is easily seen that, when G is connected, such a non-zero vector y exists only if G is bipartite, and that if G is connected and bipartite, there is a one-dimensional space of such solutions y.

Consequently, in the non-bipartite case (i), the vectors $x_1, \ldots, x_g$ are linearly independent. We apply Lemma 6.2 with $N = h$, $m = g$ and using the vectors $x_1, \ldots, x_g$. Let $\mathcal{L}$, $\mathcal{L}^\perp$ and $\mathcal{L}_0$ be as in Lemma 6.2. Then $\mathcal{L}_G^{(1)} = \mathcal{L}^\perp$, and thus $\mathcal{L}_G^{(1)}$ has rank $h - g$, by Lemma 6.2.

Furthermore, by Lemma 2.1 and (2.1),

$$\det(\mathcal{L}_0) = \left(\det(\langle x_i, x_j\rangle)_{i,j=1}^{g}\right)^{1/2} = \det(A + D_G)^{1/2}.$$

Moreover, $(t_v,\, v \in V(G))$ solves (6.1) if and only if $t_v \equiv -t_w \pmod 1$ for every edge vw. Going around an odd cycle, we see that $t_v \equiv 0$ or $t_v \equiv 1/2$ for every vertex on the cycle. Since G is connected, it follows that there are exactly two solutions to (6.1): $t_v \equiv 0$ for every v and $t_v \equiv 1/2$ for every v. Hence $q = 2$ in Lemma 6.2, and the result follows.

Now suppose that G is bipartite. Then the vectors $x_1, \ldots, x_{g-1}$ are linearly independent and x_g can be written as a $\{\pm 1\}$-combination of $x_1, \ldots, x_{g-1}$, since the sum of vectors x^v over all vertices v on either side of the vertex bipartition gives the vector $(1, 1, \ldots, 1)$. We apply Lemma 6.2 with $N = h$, $m = g - 1$, and using the vectors $x_1, \ldots, x_{g-1}$. The lemma asserts that $\mathcal{L}_G^{(1)} = \mathcal{L}^{\perp}$ has rank $h - g + 1$, and

$$\det(\mathcal{L}_0) = \left(\det(\langle x_i, x_j\rangle)_{i,j=1}^{g-1}\right)^{1/2} = \det(A' + D'_G)^{1/2}.$$

Finally, let $w \in V(G)$ correspond to x_g. If $(t_v,\, v \in V(G) \setminus \{w\})$ solves (6.1) then $t_u = 0$ for every neighbour u of w. In turn this implies that $t_u = 0$ for every vertex u at distance 2 from w, and iterating this shows that $t_u = 0$ for all vertices u in the connected graph G. Therefore $q = 1$ in Lemma 6.2 and the proof is complete. □

Example 1. When $G = K_4$,

$$\det(A + D_G) = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 48.$$

Thus Lemma 3.2(i) says that $\mathcal{L}_G^{(1)}$ has rank 2 and

$$\det(\mathcal{L}_G^{(1)}) = \frac{\sqrt{48}}{2} = \sqrt{12}.$$

Example 2. Let $G = K_2^3$ be the multigraph with two vertices and three parallel edges. Then $A + D_G = \left(\begin{smallmatrix} 3 & 3 \\ 3 & 3 \end{smallmatrix}\right)$, and deleting one row and column gives the 1×1 matrix (3). Hence $\mathcal{L}_G^{(1)}$ has rank 2 and $\det(\mathcal{L}_G^{(1)}) = \sqrt{3}$, using Lemma 3.2(ii).

We are ready to apply formula (2.7) of Theorem 2.3.

Theorem 3.3. *Suppose that G is d-regular, where $d \geqslant 3$.*

(i) *If G is non-bipartite then*

$$\begin{aligned} \mathbb{E}\, X_G &\sim \frac{2(d-1)^{dg/4}}{\sqrt{\det(A + dI)}} \left(\frac{d-1}{d(d-2)}\right)^{dg/4 - g/2} \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{gn/2} \\ &= \frac{2(d-1)^{(d-1)g/2}}{(d(d-2))^{dg/4-g/2}\sqrt{\det(A+dI)}} \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{gn/2}. \end{aligned}$$

(ii) *If G is bipartite then*

$$\mathbb{E}\,X_G \sim \frac{(d-1)^{dg/4}}{\sqrt{\det(A'+dI)}}\left(\frac{d-1}{d(d-2)}\right)^{dg/4-g/2+1/2}(2\pi n)^{1/2}\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{gn/2}$$
$$= \frac{(d-1)^{(d-1)g/2+1/2}}{(d(d-2))^{dg/4-g/2+1/2}\sqrt{\det(A'+dI)}}(2\pi n)^{1/2}\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{gn/2},$$

where A' is obtained by deleting the last row and column of A.

Proof. Let r be the rank of $\mathcal{L}_G^{(1)}$, and recall that the Hessian $H = D^2\phi(x^0)$ is diagonal and equals $-\frac{d(d-2)}{d-1}I$ by Lemma 3.1. Thus $H|_V = -\frac{d(d-2)}{d-1}I$ too, and $\det(-H|_V) = (\frac{d(d-2)}{d-1}^r)$. Hence the result follows from (3.1) and Theorem 2.3, using Lemmas 3.1 and 3.2, and the fact that $h = dg/2$. □

Example 3. For $G = K_4$, $d = 3$, $g = 4$ and thus, using Example 1,

$$\mathbb{E}\,X_G \sim \frac{2\cdot 2^4}{3\sqrt{48}}\left(\frac{4}{3}\right)^{2n} = \frac{8}{3\sqrt{3}}\left(\frac{4}{3}\right)^{2n}.$$

Example 4. For the bipartite multigraph $G = K_2^3$ with two vertices and three parallel edges we have $d = 3$, $g = 2$, and by Example 2

$$\mathbb{E}\,X_G \sim \frac{8}{3\sqrt{3}}\sqrt{\pi n}\left(\frac{4}{3}\right)^{n}.$$

4. The second moment of X_G

We now work towards an asymptotic expression for the second moment of X_G, using the same approach as in the previous section. To simplify our calculations we consider only regular multigraphs G of degree at least three.

Given a pair (M_1, M_2) of perfect matchings in $L_n(G)$, for a vertex $i \in V(G)$ and two (possibly equal) edges $e, f \ni i$, let ℓ_{ief} be the number of vertices in V_i whose incident edges in M_1 and M_2 lie, respectively, in the fibres F_e and F_f. Form these numbers into the gd^2-dimensional vector $\ell = \ell(M_1, M_2) = \big(\ell_{ief} : i \in [g],\, e, f \ni i\big)$. Let

$$V^* := \Big\{\big(z_{ief} : i \in [g],\, e, f \ni i\big) \in \mathbb{R}^{gd^2} : \quad \text{for every } e \in E(G) \text{ with endpoints } i \text{ and } j,$$
$$z_{iee} = z_{jee}, \qquad \sum_{f\ni i} z_{ief} = \sum_{f\ni j} z_{jef}, \qquad \sum_{f\ni i} z_{ife} = \sum_{f\ni j} z_{jfe}\Big\}.$$

Then the vector ℓ belongs to the set

$$Q := \Big\{(z_{ief}) \in V^* \cap \mathbb{Z}^{gd^2} : \sum_{e,f\ni i} z_{ief} = n \quad \text{for } i \in [g]\Big\}.$$

(The three conditions in V^* follow from consideration of the edges in $M_1 \cap M_2$, M_1 and M_2, respectively.) Fix a particular vector z with $nz \in Q$. (By our assumption that there is a perfect matching in $L_n(G)$, it follows that at least one such vector exists.) Then $Q = \mathcal{L}_G^{(2)} + nz$, where $\mathcal{L}_G^{(2)}$ is the lattice defined by

$$\mathcal{L}_G^{(2)} := \left\{ (v_{ief}) \in V^* \cap \mathbb{Z}^{gd^2} : \sum_{e,f \ni i} v_{ief} = 0 \quad \text{for } i \in [g] \right\}.$$

(The superscript 2 denotes the second moment.)

Given a pair (M_1, M_2) of perfect matchings and thus a vector $\ell \in Q$, we further define, for an edge $e \in E(G)$ and an endpoint i of e,

$$s_e = s_{ie}(\ell) = \sum_{f \ni i, f \neq e} \ell_{ief}, \quad t_e = t_{ie}(\ell) = \sum_{f \ni i, f \neq e} \ell_{ife}, \quad u_e = u_{ie}(\ell) = \sum_{f,f' \ni i; f,f' \neq e} \ell_{iff'};$$

these are the numbers of edges in the fibre F_e that belong to $M_1 \setminus M_2$, $M_2 \setminus M_1$ and $(M_1 \cup M_2)^c$, respectively, so they do not depend on the choice of endpoint i of e. We have, for every edge e and endpoint i,

$$s_e + t_e + u_e + \ell_{iee} = n.$$

We now calculate the expected number of pairs of perfect matchings (M_1, M_2) in $L_n(G)$ corresponding to a given non-negative integer vector $\ell = (\ell_{ief}) \in \mathcal{L}_G^{(2)} + nz$. First, partition each V_i into d^2 subsets of sizes $(\ell_{ief})_{e,f \ni i}$; this can be done in

$$\prod_{i=1}^{g} \frac{n!}{\prod_{e,f \ni i} \ell_{ief}!} = n!^g \prod_{i=1}^{g} \prod_{e,f \ni i} (\ell_{ief}!)^{-1}$$

ways. Given these partitions there are

$$s_e!\, t_e!\, u_e!\, \ell_{iee}!$$

possibilities for the fibre F_e (where i is an endpoint of e), with probability $1/n!$ each. Hence the expected number of pairs (M_1, M_2) of perfect matchings in $L_n(G)$ which correspond to the vector ℓ is given by

$$a_n(\ell) = n!^{g - dg/2} \prod_{i \in [g]} \left(\prod_{e \ni i} \left(\frac{s_e!\, t_e!\, u_e!}{\ell_{iee}!} \right)^{1/2} \prod_{f \ni i, f \neq e} \frac{1}{\ell_{ief}!} \right).$$

Thus we can write

$$\mathbb{E}(X_G^2) = \sum_{\ell \in (\mathcal{L}_G^{(2)} + nz) \cap nK} a_n(\ell), \tag{4.1}$$

where $K = [0,1]^{gd^2}$. This will allow us to apply the same arguments as used in Section 3.

We now switch to continuous variables $x \in \mathbb{R}^{gd^2}$, where x_{ief} corresponds to ℓ_{ief}/n. Define the functions $\sigma_{ie} = \sigma_{ie}(x)$, $\tau_{ie} = \tau_{ie}(x)$ and $\gamma_{ie} = \gamma_{ie}(x)$ to be continuous scaled analogues of s_{ie}, t_{ie} and u_{ie} respectively. That is,

$$\sigma_{ie} = \sum_{f \ni i, f \neq e} x_{ief}, \qquad \tau_{ie} = \sum_{f \ni i, f \neq e} x_{ife}, \qquad \gamma_{ie} = \sum_{f,f' \ni i; f,f' \neq e} x_{iff'},$$

so that $\sigma_{ie}(\ell/n) = s_{ie}(\ell)/n$ and so on. Then, applying (3.2), it follows that $a_n(\ell)$ satisfies condition (2.6) of Theorem 2.3 with

$$\begin{aligned}
b_n &= (2\pi n)^{g/2+3h/2-d^2g/2},\\
\psi(x) &= \prod_{i\in[g]}\prod_{e\ni i}\left(\frac{\sigma_{ie}\tau_{ie}\gamma_{ie}}{x_{iee}}\right)^{1/4}\prod_{f\ni i, f\neq e} x_{ief}^{-1/2},\\
\phi(x) &= \tfrac{1}{2}\sum_{i\in[g]}\sum_{e\ni i}\left(\sigma_{ie}\ln\sigma_{ie} + \tau_{ie}\ln\tau_{ie} + \gamma_{ie}\ln\gamma_{ie} - x_{iee}\ln x_{iee} - 2\sum_{f\ni i, f\neq e} x_{ief}\ln x_{ief}\right). \quad (4.2)
\end{aligned}$$

(Again, if some x_{ief}, σ_{ie}, τ_{ie} or γ_{ie} is 0, then we replace it by $1/n$ in the definition of $\psi(x)$.)

Let W be the domain defined by

$$W := \left\{(x_{ief}) \in V^* : \sum_{e,f\ni i} x_{ief} = 1 \quad \text{for } i \in [g]\right\}.$$

We conjecture that for all connected d-regular multigraphs G with no loops, the function ϕ has a unique maximum on $K \cap W$, attained at the point

$$x^0 = (1/d^2, \dots, 1/d^2).$$

Unfortunately, we have been unable to prove this, and have only been able to verify this computationally for $d = 3$. For future reference, note that

$$\psi(x^0) = \big((d-1)d^{d-2}\big)^{dg}, \qquad \phi(x^0) = g\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right). \quad (4.3)$$

One approach to finding the maximum of ϕ is to mimic the proof of Lemma 3.1. The function ϕ can be written as the sum over $i = 1, \dots, g$ of functions ϕ_i, where the sets of variables appearing in different ϕ_i are disjoint. For convenience we drop the index i and rename all variables corresponding to vertex i as $x_{ef} := x_{ief}$, and let $\sigma_e := \sigma_{ie}$, $\tau_e := \tau_{ie}$, $\gamma_e := \gamma_{ie}$. Then

$$\phi_i(x) = \tfrac{1}{2}\sum_{e\ni i}\left\{\sigma_e\ln\sigma_e + \tau_e\ln\tau_e + \gamma_e\ln\gamma_e - x_{ee}\ln x_{ee} - 2\sum_{f\ni i, f\neq e} x_{ef}\ln x_{ef}\right\}.$$

Since G is d-regular and ϕ_i depends only on the degree of i in G, all the functions ϕ_i are equivalent under relabelling of variables.

Now define the domain

$$\Sigma_{d^2} = \left\{(x_{ef})_{e,f\ni i} \in [0,1]^{d^2} : \sum_{e,f\ni i} x_{ef} = 1\right\}.$$

It suffices to prove that ϕ_i has a unique maximum on Σ_{d^2} attained at the point $(1/d^2, \dots, 1/d^2)$. Applying the Lagrange multiplier method to Σ_{d^2}, we see that at an interior maximum point, all partial derivatives of ϕ_i must be equal. This gives $d^2 - 1$ (nonlinear) equations (together with $\sum_{e,f} x_{ef} = 1$) to be solved for d^2 variables. We tried to solve this system using `Maple`. Unfortunately, `Maple` seems unable to handle the computations for $d \geqslant 4$. Hence we only have the desired result for $d = 3$.

Lemma 4.1. *If G is 3-regular then the function ϕ defined by* (4.2) *has a unique maximum on $K \cap W$ attained at the point* $(1/9, \ldots, 1/9) \in \mathbb{R}^{9g}$.

Proof. As explained above, we consider only the function ϕ_i for a fixed vertex i. Using `Maple`, we solved for points in $\{(x_{ef})_{e,f} : \sum_{e,f} x_{ef} = 1\}$ where all the 9 partial derivatives of ϕ_i are equal. Exactly four solutions were found, of which only one lies in $[0, 1]^9$, giving the point $x^0 = (1/9, \ldots, 1/9) \in \Sigma_9$. (The other three solutions each contain both positive and negative entries.) We have $\phi(x^0) = \ln(4/3)$.

It remains to consider the boundary, where one or several $x_{ef} = 0$. If $x_{ee} = 0$ and $\gamma_f > 0$ for $f \neq e$, then $\frac{\partial}{\partial x_{ee}} \phi(x) = +\infty$, and thus x is not a maximum point. Similarly, x cannot be a maximum point if $x_{ef} = 0$, where $e \neq f$ and at most one of σ_e, τ_f and $\gamma_{f'}$ (where f' is the third index) vanishes. It is easily seen that the only remaining cases are when the only non-zero variables (after relabelling the indices as $1, 2, 3$ in some order) are $\{x_{12}, x_{21}\}$, $\{x_{11}, x_{22}, x_{33}\}$ or $\{x_{11}, x_{12}, x_{13}\}$, or a subset of one of these. In the first case we have $\phi = 0$. In the two latter cases, ϕ_i equals, after relabelling, $\frac{1}{2}\phi_v$ defined in (3.6) (at the corresponding step of the first moment calculation), and thus the maximum over one of these sets is $\frac{1}{2}\ln(4/3) < \phi(x_0)$. (We omit the details.) Hence, there is no global maximum on the boundary.

Consequently, x^0 is the unique maximum point of ϕ_i on Σ_9. Arguing as in Lemma 3.1 completes the proof. □

Let $V = W - z$ be the subspace spanned by $\mathcal{L}_G^{(2)}$, *i.e.*,

$$V := \left\{ (x_{ief}) \in V^* : \sum_{e,f \ni i} x_{ief} = 0 \quad \text{for } i \in [g] \right\}.$$

Theorem 4.2. *Suppose that G is d-regular, where $d \geqslant 3$. If the function ϕ defined in* (4.2) *has a unique maximum on $K \cap W$ at $x^0 = (1/d^2, \ldots, 1/d^2)$, then*

$$\mathbb{E}(X_G^2) \sim \frac{((d-1)d^{d-2})^{dg}}{\det\left(\mathcal{L}_G^{(2)}\right) \det(-H|_V)^{1/2}} (2\pi n)^{r/2+g/2+3dg/4-d^2g/2} \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)^{gn},$$

where r is the rank of $\mathcal{L}_G^{(2)}$ and $H = D^2\phi(x^0)$ is the Hessian of ϕ at x^0, provided the determinant in the denominator is non-zero. In particular, this expression holds for all 3-regular connected graphs G.

Proof. This is now an immediate consequence of Theorem 2.3, using (4.1) and (4.3). The final statement follows from Lemma 4.1. □

It remains to calculate the determinants of $\mathcal{L}_G^{(2)}$ and $-H|_V$, and the rank r. In the non-bipartite case, part of this is covered by the next lemma.

Lemma 4.3. *Suppose that G is non-bipartite and d-regular, where $d \geqslant 3$. Recall that h denotes the number of edges in G, so $h = dg/2$. Then the lattice $\mathcal{L}_G^{(2)}$ has rank $d^2 g - (g +$

$3h) = d^2g - g - 3dg/2$ and determinant

$$\det\left(\mathcal{L}_G^{(2)}\right) = 2^{3h/2-3g/2-2}(d(d-2))^{h/2-g/2}\det(dI+A)\det(d(2d-3)I-A)^{1/2}$$
$$= 2^{3h/2-3g/2-2}(d(d-2))^{h/2-g/2}\prod_{i=1}^{g}(d+\alpha_i)(d(2d-3)-\alpha_i)^{1/2},$$

where $\alpha_1,\ldots,\alpha_g$ are the eigenvalues of A.

Proof. The linear space V spanned by $\mathcal{L}_G^{(2)}$ is the subspace of $\mathbb{R}^{gd^2}$ orthogonal to the following $g+3h$ vectors:

- one vector x^{0j} for every $j \in V(G)$, with $x^{0j}_{ief} = \mathbf{1}[i=j]$,
- one vector $x^{1\varepsilon}$ for every $\varepsilon \in E(G)$, with $x^{1\varepsilon}_{ief} = \vec{a}_{i\varepsilon}\mathbf{1}[e=f=\varepsilon]$,
- one vector $x^{2\varepsilon}$ for every $\varepsilon \in E(G)$, with $x^{2\varepsilon}_{ief} = \vec{a}_{i\varepsilon}\mathbf{1}[e=\varepsilon\neq f]$,
- one vector $x^{3\varepsilon}$ for every $\varepsilon \in E(G)$, with $x^{3\varepsilon}_{ief} = \vec{a}_{i\varepsilon}\mathbf{1}[e\neq\varepsilon=f]$.

Relabel these vectors (in this order) as $x_1,\ldots,x_{g+3h}$. Then their Gram matrix Γ can be written in block form, with blocks of dimensions g,h,h,h:

$$\Gamma = \begin{pmatrix} d^2I & \vec{A} & (d-1)\vec{A} & (d-1)\vec{A} \\ \vec{A}^T & 2I & 0 & 0 \\ (d-1)\vec{A}^T & 0 & 2(d-1)I & \vec{A}^T\vec{A}-2I \\ (d-1)\vec{A}^T & 0 & \vec{A}^T\vec{A}-2I & 2(d-1)I \end{pmatrix}.$$

In order to evaluate the Gram determinant $\det(\Gamma)$, we may make an orthogonal change of basis in the first component $\mathbb{R}^g$, and another orthogonal change of basis in each of the components $\mathbb{R}^h$ (we choose the same change in all three). It is well known that we can make such changes of basis such that any given $g\times h$ matrix B obtains the form of a diagonal $g\times g$ matrix D_s with $h-g$ additional columns of 0s; this is known as the singular value decomposition of B, and is easily seen by choosing an orthonormal basis $z_1,\ldots,z_h$ in $\mathbb{R}^h$ such that B^TB is diagonal, and then choosing an orthonormal basis in $\mathbb{R}^g$ containing the vectors $Bz_i/\|Bz_i\|$, for all i such that $Bz_i\neq 0$. We choose such bases for $B=\vec{A}$. The diagonal entries $s_1,\ldots,s_g$ of D_s can be assumed to be non-negative, and they are identified by the fact that the eigenvalues of $BB^T = \vec{A}\vec{A}^T$ are $\{s_i^2\}$. By (2.2), we thus have

$$s_i^2 = d - \alpha_i. \tag{4.4}$$

Hence, with $\tilde{D}_s = (D_s,\ 0)$ a $g\times h$ matrix with non-zero elements given by (4.4),

$$\det\Gamma = \begin{vmatrix} d^2I & \tilde{D}_s & (d-1)\tilde{D}_s & (d-1)\tilde{D}_s \\ \tilde{D}_s^T & 2I & 0 & 0 \\ (d-1)\tilde{D}_s^T & 0 & 2(d-1)I & \tilde{D}_s^T\tilde{D}_s-2I \\ (d-1)\tilde{D}_s^T & 0 & \tilde{D}_s^T\tilde{D}_s-2I & 2(d-1)I \end{vmatrix}. \tag{4.5}$$

Since D_s is a diagonal matrix, we can reorder the rows and columns in (4.5) so that we obtain a block diagonal matrix with g 4×4 blocks

$$\Gamma_i := \begin{pmatrix} d^2 & s_i & (d-1)s_i & (d-1)s_i \\ s_i & 2 & 0 & 0 \\ (d-1)s_i & 0 & 2(d-1) & s_i^2-2 \\ (d-1)s_i & 0 & s_i^2-2 & 2(d-1) \end{pmatrix} \tag{4.6}$$

and $h-g$ identical 3×3 blocks

$$\Gamma_0 := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2(d-1) & -2 \\ 0 & -2 & 2(d-1) \end{pmatrix}. \tag{4.7}$$

Hence, by straightforward calculations,

$$\begin{aligned} \det(\Gamma) &= \det(\Gamma_0)^{h-g} \prod_{i=1}^{g} \det(\Gamma_i) \\ &= (8d(d-2))^{h-g} \prod_{i=1}^{g} (2d-s_i^2)^2 \big(2d^2-4d+s_i^2\big) \\ &= (8d(d-2))^{h-g} \prod_{i=1}^{g} (d+\alpha_i)^2 (d(2d-3)-\alpha_i). \end{aligned} \tag{4.8}$$

Since G is non-bipartite, $-d < \alpha_i \leqslant d$ for every i, and thus (4.8) shows that $\det(\Gamma) \neq 0$. Hence, the vectors $x_1, \dots, x_{g+3h}$, or in different notation

$$\{x^{0j} : j \in V(G)\} \cup \{x^{1\varepsilon}, x^{2\varepsilon}, x^{3\varepsilon} : \varepsilon \in E(G)\}, \tag{4.9}$$

are linearly independent, so they form a basis in $V^{\perp}$.

We apply Lemma 6.2, with $N = d^2 g$, $m = g + 3h = g + 3dg/2$, and using the vectors $x_1, \dots, x_{g+3h}$ in (4.9). Then $\mathcal{L}_G^{(2)} = \mathcal{L}^{\perp}$. Hence, $\operatorname{rank}(\mathcal{L}_G^{(2)}) = N - m = d^2 g - g - 3h$. We have $\det(\mathcal{L}_0) = \det(\Gamma)^{1/2}$ by Lemma 2.1. Finally, we claim that there are 4 solutions (mod 1) to (6.1): if we let t_{0j} denote the coefficient of x^{0j}, and so on, the solutions have $t_{0j} = t_0$ for all j and $t_{1\varepsilon} = t_1$, $t_{2\varepsilon} = t_2$, $t_{3\varepsilon} = t_3$ for all ε, where $(t_0, t_1, t_2, t_3) = (0,0,0,0)$, $(0, 0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$, or $(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2})$. (To prove this, first consider the equations in (6.1) which correspond to variables x_{iee}, and use the existence of an odd cycle. This gives the possible values of t_0 and t_1. The rest of the proof follows by considering the equations in (6.1) which correspond to variables x_{ief} for a given vertex i, with $e \neq f$.)

Hence $q = 4$, and Lemma 6.2 yields

$$\det\big(\mathcal{L}_G^{(2)}\big) = \det(\mathcal{L}^{\perp}) = \det(\Gamma)^{1/2}/4.$$

The result follows by (4.8). □

Example 5. For $G = K_4$, we have $d = 3$, $g = 4$, $h = 6$, and A has the eigenvalues $3, -1, -1, -1$. Hence Lemma 4.3 yields $\det(\mathcal{L}_G^{(2)}) = 2^7\, 3^{5/2}\, 5^{3/2}$.

We believe that there is a similar result for regular bipartite graphs, but we have not explored it. (Presumably, the rank is then $d^2 g - g - 3h + 2$.)

Unfortunately, we have not been able to find a similar general formula for $\det(-H|_V)$ in Theorem 4.2. However, this quantity can be calculated directly for a particular graph G, once a basis for $\mathcal{L}_G^{(2)}$ is known.

Example 6. When $G = K_4$, using `Maple` we found a basis $\{z_1, \ldots, z_{14}\}$ of V and then calculated $\det(-H|_V) = 2^{-22}\, 3^{28}\, 5^{-1}\, 11^3$ using (2.5). Hence by Theorem 4.2 and Example 5,

$$\mathbb{E}\left(X_G^2\right) \sim 2^{16}\, 3^{-9/2}\, 5^{-1}\, 11^{-3/2} \left(\frac{4}{3}\right)^{4n}.$$

Example 7. When $G = K_2^3$ is the multigraph with two vertices and three parallel edges, `Maple` computations confirmed that $\mathcal{L}_G^{(2)}$ has rank 9 and gave

$$\det\left(\mathcal{L}_G^{(2)}\right) = 2^4\, 3^{3/2} \quad \text{and} \quad \det(-H|_V) = 2^{-16}\, 3^{18}\, 5^2.$$

Hence by Theorem 4.2,

$$\mathbb{E}(X_G^2) \sim 2^{11}\, 3^{-9/2}\, 5^{-1}\, \pi n \left(\frac{4}{3}\right)^{2n}.$$

5. Short cycles in random lifts

Let Z_k denote the number of cycles of length k in $L_n(G)$, for $k \geqslant 2$. (Note that Z_2 is zero unless there are multiple edges in G.) To apply the small subgraph conditioning method to X_G, we must understand the distribution of short cycles in random lifts, as well as their interaction with perfect matchings. This will enable us to verify conditions (A1)–(A3) of [11, Theorem 9.12], with their Y_n given by our X_G (the index n is suppressed), and with their X_{kn} given by our Z_k.

To compute the limiting distributions in (A1) and (A2) of [11, Theorem 9.12], we will use the method of moments. Moreover, for (A2) we will be guided by [11, Lemma 9.17 and Remark 9.18], which tell us that we need only compute asymptotically

$$\mathbb{E}(X_G\,(Z_2)_{j_2} \cdots (Z_m)_{j_m}) / \,\mathbb{E}\, X_G,$$

for integer constants $m \geqslant 0$ and $j_2, \ldots, j_m \geqslant 0$. Here $(Z)_j$ denotes the falling factorial $Z(Z-1)\cdots(Z-j+1)$.

Let k be a fixed positive integer. It is more convenient to count rooted oriented k-cycles, which introduces a factor of $2k$ into the calculations. A k-cycle in $L_n(G)$ can then be thought of as a lift of a *non-backtracking* closed k-walk in G, which is a walk $i_0 e_1 i_1 e_2 \cdots i_{k-1} e_k$ in G such that e_j is an edge of G with endpoints $\{i_j, i_{j+1}\}$ and $e_j \neq e_{j-1}$, for $1 \leqslant j \leqslant k$. (Here and throughout this section, arithmetic on indices in k-walks is performed modulo k.) Note that if G is simple then any three consecutive vertices on the walk must all be distinct. These walks arise in various contexts (see, for example, [1, 5, 10]) and have also been called *irreducible* [9] and *non-backscattering* [13]. Denote by w_k the number of non-backtracking closed k-walks in G, for $k \geqslant 2$.

The following lemma shows that condition (A1) of [11, Theorem 9.12] holds.

Lemma 5.1. *Let $\lambda_k = w_k/(2k)$ for all $k \geqslant 2$, where w_k is the number of non-backtracking closed k-walks in G. Then $Z_k \sim \mathrm{Po}(\lambda_k)$, jointly for all $k \geqslant 2$.*

Proof. Fix a non-backtracking closed k-walk $C = i_0e_1i_1\cdots i_{k-1}e_k$ in G. The (oriented) k-cycle $C' = f_1f_2\cdots f_k$ in $L_n(G)$ is a lift of C if $f_j \in F_{e_j}$ for $j = 1,\dots,k$. Hence the number of possible lifts C' of C is $(1+o(1))n^k$, and each will appear in $L_n(G)$ with probability $(1+o(1))n^{-k}$. It follows that

$$\mathbb{E}\, Z_k = \sum_C \sum_{C'} \mathbb{P}(C' \subset L_n(G)) = \frac{w_k}{2k} + o(1).$$

Similar arguments hold for higher joint factorial moments, completing the proof. □

For the remainder of this section we restrict our attention to d-regular multigraphs with $d \geqslant 3$. Next we verify condition (A2) of [11, Theorem 9.12] using the approach suggested in [11, Remark 9.18].

Lemma 5.2. *Suppose that G is d-regular with $d \geqslant 3$, and for $k \geqslant 2$, let*

$$\mu_k = \left(1 + \left(\frac{-1}{d-1}\right)^k\right)\lambda_k.$$

Then, for any integer $m \geqslant 2$ and non-negative integers $j_2,\dots,j_m$,

$$\frac{\mathbb{E}(X_G\,(Z_2)_{j_2}\cdots(Z_m)_{j_m})}{\mathbb{E}\,X_G} \longrightarrow \prod_{i=2}^{m} \mu_i^{j_i} \quad \text{as } n \to \infty.$$

Proof. For ease of notation, throughout this proof we write $\mathbb{P}(M) := \mathbb{P}(M \subseteq L_n(G))$, $\mathbb{P}(M, C') := \mathbb{P}(M \subseteq L_n(G), C' \subseteq L_n(G))$, and so on. First we estimate $\mathbb{E}(X_G\,Z_k)$. We write

$$\mathbb{E}(X_G\,Z_k) = \sum_M \sum_C \sum_{C'} \mathbb{P}(M, C') = \sum_M \mathbb{P}(M) \sum_C \sum_{C'} \mathbb{P}(C'|M),$$

where the sums extend over all possible perfect matchings M in $L_n(G)$, all non-backtracking closed k-walks C in G, and all their possible lifts C', respectively.

To calculate the inner double sum, we fix a perfect matching M_0 and condition on its presence in $L_n(G)$. Let $C = i_0e_1i_1\cdots i_{k-1}e_k$ be a given non-backtracking closed k-walk in G. For a lift C' of C with edges $f_1f_2\cdots f_k$, let

$$\xi_j(C') = \begin{cases} 1 & \text{if } f_j \in M_0, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } 1 \leqslant j \leqslant k.$$

To estimate the expected number of lifts of C given M_0, we break the sum over all C' according to the vector $\xi(C')$:

$$\sum_{C'} \mathbb{P}(C'|M_0) = \sum_{u \in \{0,1\}^k} \sum_{C' : \xi(C') = u} \mathbb{P}(C'|M_0).$$

Let ℓ_e be the number of edges of M_0 in the fibre F_e, and say that M_0 is *good* if

$$|\ell_e - n/d| \leqslant n^{2/3} \quad \text{for every } e.$$

We may assume that M_0 is good, since the calculations for the expectation in Section 3 show that the contribution from other matchings is negligible. (Specifically, this follows from the proof of Lemma 6.3: in particular the fact that $S_2 = o(1)$, $S_3 = o(1)$, using notation from that proof.)

Hence, for a given $u = (u_1, u_2, \dots, u_k) \in \{0,1\}^k$,

$$\mathbb{P}(C'|M_0) \sim \left(\frac{1}{n - n/d}\right)^{k - \sum_i u_i}.$$

Let $t_{00}(u)$ and $t_{01}(u)$ be the numbers of substrings 00 and 01 in u, respectively. Next we prove that the number of lifts $C' = f_1 \cdots f_k$ of C such that $\xi(C') = u$ is asymptotically equal to

$$\left(n - \frac{2n}{d}\right)^{t_{00}(u)} \left(\frac{n}{d}\right)^{t_{01}(u)}.$$

Indeed, let V_{ie} be the set of endpoints in V_i of the ℓ_e edges in $M_0 \cap F_e$, for i incident to $e \in E(G)$. If, say, $u_1 = u_2 = 0$, which means that neither f_1 nor f_2 are in M_0, then we can choose the end of f_1 in V_{i_1} from $V_{i_1} \setminus (V_{i_1e_1} \cup V_{i_1e_2})$, and $|V_{i_1} \setminus (V_{i_1e_1} \cup V_{i_1e_2})| \sim n - 2n/d$ since we assume that M_0 is good. Similarly, if $u_1 = 0$ and $u_2 = 1$, which means that $f_1 \notin M_0$ but $f_2 \in M_0$, then we have to choose the end of f_1 from $V_{i_1e_2}$, a set of size $\sim n/d$. Note also that if $u_1 = 1$ then we must have $u_2 = 0$, and if we have already selected the end w of f_1 in V_{i_0}, then the other end of f_1 is completely determined as the partner of w in M_0.

Multiplying these two expressions together yields that

$$\sum_{C' : \xi(C') = u} \mathbb{P}(C'|M_0) = b_{u_1u_2} \cdots b_{u_{k-1}u_k} b_{u_ku_1} + o(1),$$

where $b_{00}, b_{01}, b_{10}, b_{11}$ form the matrix

$$B = \begin{pmatrix} \frac{d-2}{d-1} & \frac{1}{d-1} \\ 1 & 0 \end{pmatrix}.$$

Note that B has eigenvalues 1 and $-1/(d-1)$. Summing over all $u = (u_1, \dots, u_k)$, we find that the conditional expected number of lifts of C is

$$\sum_{C'} \mathbb{P}(C'|M_0) = \mathrm{Tr}(B^k) + o(1) = 1 + \left(\frac{-1}{d-1}\right)^k + o(1).$$

Hence the expected number of k-cycles in $L_n(G)$, conditioned on the existence of a given good perfect matching M_0, is asymptotically equal to

$$\sum_C \sum_{C'} \mathbb{P}(C'|M_0) \sim \mu_k := \left(1 + \left(\frac{-1}{d-1}\right)^k\right) \frac{w_k}{2k} = \left(1 + \left(\frac{-1}{d-1}\right)^k\right) \lambda_k.$$

Finally,

$$\mathbb{E}(X_G Z_k) \sim \sum_M \mathbb{P}(M) \mu_k = \mu_k \, \mathbb{E} X_G.$$

All the above calculations work similarly for higher factorial moments and yield the desired result. □

Denote a directed edge of G by (e, i, j), where $e \in E(G)$ is incident to $i, j \in V(G)$ and $i \neq j$; this denotes e directed from i to j. Now let R be the $dg \times dg$ matrix with rows and columns indexed by directed edges of G, and

$$R_{(e,i,j),(f,p,q)} = \begin{cases} 1 & \text{if } p = j \text{ and } f \neq e, \\ 0 & \text{otherwise.} \end{cases}$$

(Here R is the adjacency matrix of a version of the directed line graph of G, where U-turns are forbidden.) Then

$$w_k = \operatorname{Tr}(R^k) = \theta_1^k + \cdots + \theta_{dg}^k, \tag{5.1}$$

where $\theta_1, \ldots, \theta_{dg}$ are the eigenvalues of R. Note that $d-1$ is an eigenvalue of R with eigenvector $(1, 1, \ldots, 1)^T$; since R has non-negative entries, this is the eigenvalue with largest modulus. Now for $k \geqslant 2$, the quantity μ_k defined in Lemma 5.2 equals

$$\mu_k = (1 + \delta_k)\lambda_k, \quad \text{where } \delta_k = \left(\frac{-1}{d-1}\right)^k > -1.$$

Therefore the quantity $\sum_k \lambda_k \delta_k^2$ in condition (A3) of [11, Theorem 9.12] is

$$\begin{aligned} \sum_k \lambda_k \delta_k^2 &= \sum_{k \geqslant 1} \frac{w_k}{2k\,(d-1)^{2k}} = \sum_{k \geqslant 1} \frac{1}{2k} \sum_{t=1}^{dg} \left(\frac{\theta_t}{(d-1)^2}\right)^k \\ &= -\frac{1}{2} \sum_{t=1}^{dg} \ln\left(1 - \frac{\theta_t}{(d-1)^2}\right), \end{aligned}$$

which is finite as required. Furthermore,

$$\begin{aligned} \exp\left(\sum_k \lambda_k \delta_k^2\right) &= (d-1)^{dg} \left(\prod_{t=1}^{dg} ((d-1)^2 - \theta_t)\right)^{-1/2} \\ &= (d-1)^{dg} \det\big((d-1)^2 I - R\big)^{-1/2}. \end{aligned} \tag{5.2}$$

In order to assist with the verification of condition (A4) from from [11, Theorem 9.12], we will rewrite this expression in terms of the adjacency matrix A of G. The following result was proved by Friedman [9].

Lemma 5.3 ([9], Theorem 10.3). *Suppose that G is d-regular with $d \geqslant 3$ and let $\alpha_1, \ldots, \alpha_g$ be the eigenvalues of the adjacency matrix of G. For $i = 1, \ldots, g$, let β_i^+ and β_i^- denote the roots of the quadratic $x^2 - \alpha_i x + d - 1 = 0$. That is,*

$$\beta_i^+ = \tfrac{1}{2}\alpha_i + \sqrt{\tfrac{1}{4}\alpha_i^2 - (d-1)}, \quad \beta_i^- = \tfrac{1}{2}\alpha_i - \sqrt{\tfrac{1}{4}\alpha_i^2 - (d-1)}.$$

Then the eigenvalues of R are β_i^+, β_i^- for $i = 1, \ldots, g$, together with 1 and -1, the latter two repeated $g(d-2)/2$ times each. Hence, for $k \geqslant 2$, the number of non-backtracking closed *k-walks in G is given by*

$$w_k = \tfrac{1}{2}g(d-2)\big(1+(-1)^k\big) + \sum_{i=1}^{g}\big((\beta_i^+)^k + (\beta_i^-)^k\big).$$

Note that there may be repetitions among β_i^+, β_i^-, and some of these may coincide with ± 1. Hence the multiplicities of these eigenvalues may not be exactly 1 or $g(d-2)/2$: see Example 8 below.

We now use Lemma 5.3 to rewrite (5.2) in terms of the eigenvalues of the adjacency matrix of G.

Corollary 5.4. *Suppose that G is d-regular, with $d \geqslant 3$. The expression in* (5.2) *can be written as*

$$\begin{aligned}
&\exp\Big(\sum_k \lambda_k \delta_k^2\Big) \\
&\quad = (d-1)^{dg-g/2}((d-1)^4-1)^{-(d-2)g/4}\ \det((d-1)^3+1)I-(d-1)A)^{-1/2} \\
&\quad = (d-1)^{dg-g/2}((d-1)^4-1)^{-(d-2)g/4}\ \prod_{i=1}^{g}\big((d-1)^3+1-(d-1)\alpha_i\big)^{-1/2}.
\end{aligned}$$

Proof. It follows from Lemma 5.3 that the characteristic polynomial of R is given by

$$\begin{aligned}
\det(\lambda I - R) &= \prod_{i=1}^{dg}(\lambda-\theta_i) = (\lambda-1)^{(d-2)g/2}(\lambda+1)^{(d-2)g/2}\prod_{i=1}^{g}(\lambda-\beta_i^+)(\lambda-\beta_i^-) \\
&= (\lambda^2-1)^{(d-2)g/2}\prod_{i=1}^{g}(\lambda^2-\alpha_i\lambda+d-1) \\
&= (\lambda^2-1)^{(d-2)g/2}\ \det((\lambda^2+d-1)I-\lambda A).
\end{aligned}$$

The proof is completed by substituting this into (5.2) with $\lambda = (d-1)^2$. $\square$

Example 8. When $G = K_4$ the eigenvalues of A are $\alpha_1 = 3, \alpha_2 = \alpha_3 = \alpha_4 = -1$. By Lemma 5.3, the eigenvalues of R are 2, 1 (three times), -1 (twice), and $\frac{1}{2}(-1 \pm \sqrt{7}i)$ (three times each), so the number of non-backtracking closed k-walks in K_4 is

$$w_k = 2^k + 3 + 2(-1)^k + 3\left(\frac{-1+\sqrt{7}i}{2}\right)^k + 3\left(\frac{-1-\sqrt{7}i}{2}\right)^k.$$

Furthermore, by Corollary 5.4,

$$\exp\Big(\sum_k \lambda_k \delta_k^2\Big) = 2^{10}\,15^{-1}\det(9I-2A)^{-1/2} = 2^{10}\,3^{-3/2}\,5^{-1}\,11^{-3/2}.$$

Example 9. The multigraph with two vertices connected by d parallel edges has adjacency matrix

$$A = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}.$$

We have $\beta_1^{\pm}, \beta_2^{\pm} = \pm(d-1), \pm 1$ and by Lemma 5.3, the matrix R has eigenvalues $\pm(d-1)$ and ± 1, the latter with multiplicities $d-1$. Hence $w_k = 2(d-1)^k + 2(d-1)$ if $k \geqslant 2$ is even, and $w_k = 0$ if k is odd. Corollary 5.4 yields, after some algebra,

$$\exp\left(\sum_k \lambda_k \delta_k^2\right) = (d-1)^{2d-1} d^{-d/2} (d-2)^{-d/2} (d^2 - 2d + 2)^{-d/2+1/2}.$$

For example, when $d = 3$ this is $2^5 3^{-3/2} 5^{-1}$, while for $d = 4$ it is $2^{-15/2} 3^7 5^{-3/2}$.

To complete this section, we prove a concentration result for the number of perfect matchings in $L_n(G)$ when $G = K_4$ and when G is the multigraph K_2^3 with 2 vertices and 3 parallel edges. We conjecture that the analogous result is true for any connected d-regular multigraph G with no loops, where $d \geqslant 3$, with $\delta_k = -(1/(d-1))^k$.

Corollary 5.5. *For $k \geqslant 3$, let w_k be the number of non-backtracking closed walks of length k in K_4, and define $\lambda_k = w_k/2k$. Further, let Y_k be a Poisson random variable with expectation λ_k, with $\{Y_k\}_k$ independent, and define $\delta_k = (-1/2)^k$. Then, with $G = K_4$,*

$$\frac{X_G}{\mathbb{E}\, X_G} \xrightarrow{\mathrm{d}} W := \prod_{i=3}^{\infty} (1+\delta_i)^{Y_i} e^{-\lambda_i \delta_i}.$$

Proof. Let $X = X_{K_4}$. It follows from Examples 3 and 6 that

$$\frac{\mathbb{E}(X^2)}{(\mathbb{E}\, X)^2} \sim 2^{10}\, 3^{-3/2}\, 5^{-1}\, 11^{-3/2}.$$

By comparing with Example 8, we find that (A4) of [11, Theorem 9.12] is satisfied: that is,

$$\frac{\mathbb{E}\, X^2}{(\mathbb{E}\, X)^2} \to \exp\left(\sum_k \lambda_k \delta_k^2\right) \quad \text{as } n \to \infty.$$

The other conditions of [11, Theorem 9.12] hold, as follows from Lemmas 5.1 and 5.2. Applying [11, Theorem 9.12] completes the proof. □

The same argument applies for the multigraph with two vertices and three parallel edges, this time using Examples 4, 7 and 9, leading to the following.

Corollary 5.6. *Recall that K_2^3 denotes the multigraph with two vertices and three parallel edges. For $k \geqslant 2$, let w_k be the number of non-backtracking closed walks of length k, and define $\lambda_k = w_k/2k$. Further, let Y_k be a Poisson random variable with expectation λ_k, with*

$\{Y_k\}_k$ independent, and define $\delta_k = (-1/2)^k$. Then, with $G = K_2^3$,

$$\frac{X_G}{\mathbb{E}\, X_G} \xrightarrow{\mathrm{d}} W := \prod_{i=1}^{\infty}(1+\delta_{2i})^{Y_{2i}} e^{-\lambda_{2i}\delta_{2i}}.$$

It is immediate that the limiting distribution W satisfies $W > 0$ (with probability 1) in both Corollary 5.5 and Corollary 5.6. Hence $L_n(G)$ a.a.s. has a perfect matching, for both $G = K_4$ and $G = K_2^3$. This also follows from [12].

6. Summation by Laplace's method

In this section we prove our main approximation tool, Theorem 2.3, which performs a summation over lattice points. We will require a little more theory about lattices. The following surprising duality was proved by McMullen [14]. (See also [19].)

Lemma 6.1. *Let V be a subspace of $\mathbb{R}^N$ and let $V^\perp$ be its orthogonal complement. Let $\mathcal{L}$ and $\mathcal{L}^\perp$ be the lattices $V \cap \mathbb{Z}^N$ and $V^\perp \cap \mathbb{Z}^N$, and assume that the rank of $\mathcal{L}$ equals the dimension of V (i.e., that $\mathcal{L}$ spans V). Then $\mathcal{L}^\perp$ has rank $\dim(V^\perp) = N - \dim(V)$ and*

$$\det(\mathcal{L}^\perp) = \det(\mathcal{L}).$$

For our purposes we need a simple extension.

Lemma 6.2. *Let $0 \leqslant m \leqslant N$. Let $x_1, \dots, x_m$ be linearly independent vectors in $\mathbb{Z}^N$. Let V be the subspace of $\mathbb{R}^N$ spanned by $x_1, \dots, x_m$ and let $V^\perp$ be its orthogonal complement; thus*

$$V^\perp = \{y \in \mathbb{R}^N : \langle y, x_i \rangle = 0 \quad \text{for } i = 1, \dots, m\}.$$

Let $\mathcal{L}$ and $\mathcal{L}^\perp$ be the lattices $V \cap \mathbb{Z}^N$ and $V^\perp \cap \mathbb{Z}^N$, and let $\mathcal{L}_0$ be the lattice spanned by $x_1, \dots, x_m$ (i.e., the set $\{\sum_{i=1}^m n_i x_i : n_i \in \mathbb{Z}\}$ of integer combinations). Then $\mathcal{L}^\perp$ has rank $N - m$ and

$$\det(\mathcal{L}^\perp) = \det(\mathcal{L}) = \det(\mathcal{L}_0)/q,$$

where q is the order of the finite group $\mathcal{L}/\mathcal{L}_0$. Explicitly, q is the number of solutions $(t_1, \dots, t_m)$ in $(\mathbb{R}/\mathbb{Z})^m$ (or $(\mathbb{Q}/\mathbb{Z})^m$) of the system

$$\sum_i x_{ij} t_i \equiv 0 \pmod 1, \quad j = 1, \dots, N, \tag{6.1}$$

where $x_i = (x_{ij})_{j=1}^N$ for $i = 1, \dots, m$.

Proof. Since $\operatorname{rank}(\mathcal{L}) = m = \dim(V)$, we can apply Lemma 6.1 and conclude that

$$\operatorname{rank}(\mathcal{L}^\perp) = N - m \quad \text{and} \quad \det(\mathcal{L}^\perp) = \det(\mathcal{L}).$$

Next, $\mathcal{L}_0 \subseteq V \cap \mathbb{Z}^N = \mathcal{L}$; moreover, $\mathcal{L}_0$ and $\mathcal{L}$ both span V and thus have the same rank. Hence Lemma 2.2 shows that $\mathcal{L}/\mathcal{L}_0$ is finite and $\det(\mathcal{L}) = \det(\mathcal{L}_0)/q$. Note further that $\mathcal{L} \subseteq V = \{\sum_i t_i x_i : t_i \in \mathbb{R}\}$ and thus

$$q = |\mathcal{L}/\mathcal{L}_0| = \left|\left\{(t_i) \in [0,1)^m : \sum_i t_i x_i \in \mathcal{L}\right\}\right|.$$

Furthermore,

$$\sum_i t_i x_i \in \mathcal{L} \iff \sum_i t_i x_i \in \mathbb{Z}^N \iff \sum_i x_{ij} t_i \equiv 0 \pmod 1 \quad \text{for } j = 1, \dots, J,$$

and the characterization of q follows. □

The proof of Theorem 2.3 involves reduction to a special case, which we prove first.

Lemma 6.3. *Suppose the following.*

(i) $\mathcal{L} \subset \mathbb{R}^r$ *is a lattice with full rank* r.
(ii) $K \subset \mathbb{R}^r$ *is a compact convex set with non-empty interior* K°.
(iii) $\phi : K \to \mathbb{R}$ *is a continuous function with a unique maximum at some interior point* $x_0 \in K^\circ$.
(iv) ϕ *is twice continuously differentiable in a neighbourhood of* x_0 *and the Hessian* $H := D^2\phi(x_0)$ *is strictly negative definite.*
(v) $\psi : K_1 \to \mathbb{R}$ *is a continuous function on some neighbourhood* $K_1 \subseteq K$ *of* x_0 *with* $\psi(x_0) > 0$.
(vi) *For each positive integer* n *there is a vector* $\ell_n \in \mathbb{R}^r$.
(vii) *For each positive integer* n *there is a positive real number* b_n *and a function* $a_n : (\mathcal{L} + \ell_n) \cap nK \to \mathbb{R}$ *such that, as* $n \to \infty$,

$$a_n(\ell) = O\big(b_n e^{n\phi(\ell/n)+o(n)}\big), \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK, \tag{6.2}$$

and

$$a_n(\ell) = b_n\big(\psi(\ell/n) + o(1)\big)e^{n\phi(\ell/n)}, \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK_1, \tag{6.3}$$

uniformly for ℓ *in the indicated sets.*

Then, as $n \to \infty$,

$$\sum_{\ell \in (\mathcal{L}+\ell_n)\cap nK} a_n(\ell) \sim \frac{(2\pi)^{r/2}\psi(x_0)}{\det(\mathcal{L})\det\big(-H\big)^{1/2}}\, b_n n^{r/2} e^{n\phi(x_0)}. \tag{6.4}$$

Proof. We begin with a few simplifications. We may obviously assume that $b_n = 1$. Furthermore, by subtracting $\phi(x_0)$ from ϕ, and dividing $a_n(\ell)$ by $e^{n\phi(x_0)}$, we may suppose that $\phi(x_0) = 0$.

Since x_0 is an interior maximum point, the gradient $D\phi(x_0)$ vanishes, and a Taylor expansion at x_0 shows that, using (iv), as $|x - x_0| \to 0$,

$$\begin{aligned}\phi(x) &= \tfrac{1}{2}\langle x - x_0, D^2\phi(x_0)(x - x_0)\rangle + o(|x - x_0|^2) \\ &\leqslant -c_1|x - x_0|^2 + o(|x - x_0|^2)\end{aligned} \tag{6.5}$$

for some positive constant c_1. Consequently, there exists $\delta > 0$ such that the neighbourhood $\{x : |x - x_0| \leqslant \delta\}$ is contained in K_1 and

$$\phi(x) \leqslant -c_2|x - x_0|^2, \qquad |x - x_0| < \delta \tag{6.6}$$

for some positive constant c_2. We divide the sum in (6.4) into three parts:

$$S_1 := \sum_{|\ell/n - x_0| < n^{-1/3}}, \qquad S_2 := \sum_{n^{-1/3} \leqslant |\ell/n - x_0| < \delta}, \qquad S_3 := \sum_{|\ell/n - x_0| \geqslant \delta}.$$

In the sum S_2 we use (6.3) and (6.6); thus each term is

$$a_n(\ell) = O(e^{n\phi(\ell/n)}) = O(e^{-c_2 n^{1/3}}).$$

Since the number of terms is $O(n^r)$, we obtain $S_2 = o(1)$.

Similarly, by compactness, if $|x - x_0| \geqslant \delta$, then $\phi(x) \leqslant -c_3$ for some positive constant c_3. Consequently, for large n, (6.2) shows that each term in S_3 is

$$a_n(\ell) = O(e^{n\phi(\ell/n) + c_3 n/2}) = O(e^{-c_3 n/2}).$$

Again, the number of terms is $O(n^r)$ and we obtain $S_3 = o(1)$.

We convert the sum S_1 into an integral by picking a unit cell U of the lattice $\mathcal{L}$ and defining $a_n(y) := a_n(\ell)$ for $y \in U + \ell$, $\ell \in \mathcal{L} + \ell_n$. Let $Q_n := \bigcup_{|\ell/n - x_0| < n^{-1/3}} (U + \ell)$, and let $\widetilde{Q}_n := \{z : nx_0 + \sqrt{n}z \in Q_n\}$. Then

$$S_1 = \det(\mathcal{L})^{-1} \int_{Q_n} a_n(y)\,\mathrm{d}y = \det(\mathcal{L})^{-1} n^{r/2} \int_{\widetilde{Q}_n} a_n\big(nx_0 + \sqrt{n}z\big)\,\mathrm{d}z. \tag{6.7}$$

Note that Q_n is roughly a ball of radius $n^{2/3}$ centred at nx_0, and $\widetilde{Q}_n$ is roughly a ball of radius $n^{1/6}$ centred at 0.

If $y \in Q_n$, then $|y/n - x_0| \leqslant n^{-1/3} + O(n^{-1})$. Since the gradient $D\phi(x_0) = 0$, (iv) implies that, for $x \in Q_n/n$,

$$|D\phi(x)| = O(|x - x_0|) = O(n^{-1/3}). \tag{6.8}$$

If $y \in U + \ell \subset Q_n$, then $|y/n - \ell/n| = O(1/n)$, and (6.8) implies

$$n\phi(y/n) - n\phi(\ell/n) = O\big(nn^{-1/3}n^{-1}\big) = O\big(n^{-1/3}\big),$$

and thus (6.3) implies, uniformly for $y \in Q_n$,

$$a_n(y) = a_n(\ell) = \big(\psi(y/n) + o(1)\big)e^{n\phi(y/n)}. \tag{6.9}$$

For every fixed $z \in \mathbb{R}^r$, this and the Taylor expansion (6.5) show that, as $n \to \infty$, using the continuity of ψ,

$$a_n(nx_0 + \sqrt{n}z) \to \psi(x_0)e^{\frac{1}{2}\langle z, D^2\phi(x_0)z\rangle}.$$

Moreover, (6.6) and (6.9) provide a uniform bound, for all $z \in \mathbb{R}^r$,

$$|a_n(nx_0 + \sqrt{n}z)\mathbf{1}_{\widetilde{Q}_n}(z)| \leqslant C_1 e^{-c_2|z|^2}.$$

Further, $\mathbf{1}_{\widetilde{Q}_n}(z) \to 1$ for every z. Hence, dominated convergence shows that

$$\int_{\widetilde{Q}_n} a_n(nx_0 + \sqrt{n}z)\,\mathrm{d}z \to \int_{\mathbb{R}^r} \psi(x_0) e^{\frac{1}{2}\langle z, D^2\phi(x_0)z\rangle}\,\mathrm{d}z$$
$$= \psi(x_0)(2\pi)^{r/2} \det\bigl(-D^2\phi(x_0)\bigr)^{-1/2}.$$

The result follows from this and (6.7), together with the estimates $S_2 = o(1)$ and $S_3 = o(1)$ above. □

Proof of Theorem 2.3. First, replacing K by $K - w$, $a_n(\ell)$ by $a_n'(\ell) := a_n(\ell + nw)$, ℓ_n by $\ell_n - nw$, and translating ϕ and ψ, we reduce to the case $w = 0$ and thus $W = V$ and $\ell_n \in V$.

Choose a lattice basis $\{z_1, \dots, z_r\}$ of $\mathcal{L}$. Consider the mapping $T : \mathbb{R}^r \to V \subseteq \mathbb{R}^N$ given by $(y_1, \dots, y_r) \mapsto \sum_{i=1}^r y_i z_i$, which thus maps $\mathbb{Z}^r$ onto $\mathcal{L}$. We apply Lemma 6.3 to $\mathcal{L}' := \mathbb{Z}^r$, $K' := T^{-1}(K)$, $\phi \circ T$, $\psi \circ T$, $\ell_n' := T^{-1}(\ell_n)$, and $a_n(T(k))$, $k \in (\mathcal{L}' + \ell_n') \cap nK'$. The Hessian $D^2(\phi \circ T)(T^{-1}x_0)$ equals $\bigl(H(z_i, z_j)\bigr)_{i,j=1}^r$, and its negative has determinant, by (2.5) and (2.3),

$$\det\bigl(-H(z_i, z_j)\bigr)_{i,j=1}^r = \det(-H|_V)\det(\langle z_i, z_j\rangle)_{i,j=1}^r = \det(-H|_V)\det(\mathcal{L})^2. \tag{6.10}$$

Hence, (2.7) follows from Lemma 6.3. Note that the Hessian $D^2(\phi \circ T)(T^{-1}x_0)$ is always negative semi-definite, because x_0 is a maximum point. Hence, it is negative definite unless its determinant is zero, which is ruled out by (6.10) and the assumption that $\det(-H|_V) \neq 0$. □

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