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An Irregular Mind
(Szemerédi is 70)
pp. 1–30.

Dirac-type Questions for Hypergraphs – a Survey (or More Problems for Endre to Solve)

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Dedicated to Endre Szemerédi on the occasion of his 70th birthday

In 1952 Dirac [8] proved a celebrated theorem stating that if the minimum degree $\delta(G)$ in a graph G is at least $n/2$ then G contains a Hamiltonian cycle. In 1999, Katona and Kierstead initiated a new stream of research devoted to studying similar questions for hypergraphs, and subsequently, for perfect matchings. A pivotal role in achieving some of the most important results in both these areas was played by Endre Szemerédi. In this survey we present the current state-of-art and pose some open problems.

1. Introduction

A *k-uniform hypergraph,* or *k-graph* for short, is a pair $H=(V,E)$, where $V:=V(H)$ is a finite set of vertices and $E:=E(H)\subseteq\binom{V}{k}$ is a family of k-element subsets of V. Whenever convenient we will identify H with $E(H)$. *A matching* in H is a set of disjoint edges of H, and a matching containing all vertices of H is called *perfect.*

There are several notions of a hypercycle. Berge [2] defined a hypercycle of length m in a hypergraph H as an alternating sequence of m vertices and m edges $x_1, e_1, x_2, e_2, \dots, x_m, e_m, x_1$ such that $\{x_i, x_{i+1}\}\subseteq e_i$ for all

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$i = 1, 2, \ldots, m$, where $x_{m+1} = x_1$. Observe that there may be other vertices than $x_1, \ldots, x_m$ in the edges of a Berge cycle and that there are several nonisomorphic Berge hypercycles of length m. Bermond et al. [3] studied the existence of Hamiltonian Berge cycles under some degree conditions. Also, more recently, there has been some interest in Ramsey-type questions for Hamiltonian Berge cycles (see, e.g., [10].)

However, following the paper by Katona and Kierstead [13], another notion of a hypergraph cycle has become gradually more and more popular.

Definition 1.1. For $0 \le l \le k-1$ *a (k,l)-cycle* is a k-graph whose vertices can be ordered cyclically in such a way that the edges are segments of that cyclic order and every two consecutive edges share exactly l vertices (see Figure 1 [1]). *A Hamiltonian l-cycle* in a k-graph H is then defined as a (k,l)-cycle in H containing all vertices of H.

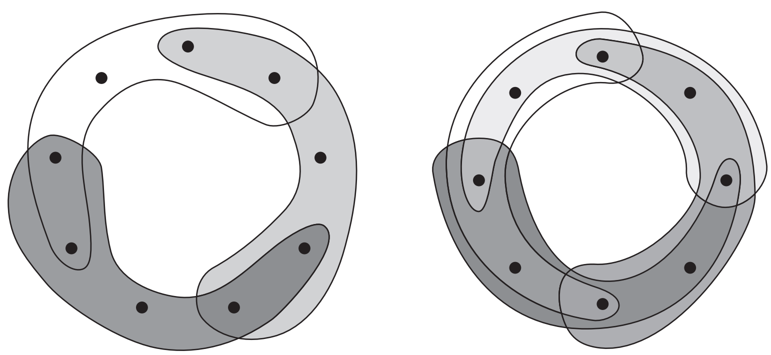

Fig. 1. A $(5,2)$-cycle and a $(5,3)$-cycle

The notion of a (k,l)-cycle, unlike the Berge hypercycle, is unique up to isomorphism. Let us denote by $C_s^{k,l}$ the (k,l)-cycle on s vertices. Observe that s must be divisible by $k-l$ and the cycle has $s/(k-l)$ edges. Furthermore, if we write $k = t(k-l)+r$, where $1 \le t \le k$ and $0 \le r \le k-l-1$ are uniquely determined by k and l, then $s/(k-l) \ge t+1$. In particular, $s \ge k+1$ for $l = k-1$ while $s \ge 2(k-l)$ for $l < k/2$.

If, in addition, $k-l$ divides k then a (k,l)-cycle is regular of degree $k/(k-l)$. Otherwise, its minimum degree is $\lfloor k/(k-l) \rfloor$ and maximum degree is $\lceil k/(k-l) \rceil$. Note also that for $l = 0$ an l-cycle reduces to a matching.

[1] All figures prepared electronically by Emory students, Domingos Dellamonica Jr. and Sangjune Lee

Given a k-graph H and d vertices $v_1, \dots, v_d \in V(H)$, $1 \le d \le k-1$, we denote by $\deg_H(v_1, \dots, v_d)$ the degree of the d-tuple $\{v_1, \dots, v_d\}$ in H, that is, the number of edges of H which contain $v_1, \dots, v_d$. For a vertex $v \in V(H)$, let $H(v)$ denote *the link of v in H* that is,

$$H(v) = \left\{ e \in \binom{V \setminus \{v\}}{k-1} \,:\, e \cup \{v\} \in H \right\}.$$

In particular, $\big|H(v)\big| = \deg_H(v)$.

Further, let

$$\delta_d(H) := \delta_d = \min\big\{ \deg_H(v_1, \dots, v_d) \,:\, \{v_1, \dots, v_d\} \subset V(H) \big\}.$$

For $d = 1$, $\delta_d(H)$ is the ordinary minimum vertex degree in H. Observe that $\delta_d(H) \le \binom{n-d}{k-d}$.

Definition 1.2. Let d, k, l, and n satisfy $1 \le d \le k-1$ and $k-l$ divide n. We define $h_d^l(k,n)$ to be the smallest integer h such that every n-vertex k-graph H satisfying $\delta_d(H) \ge h$ contains a Hamiltonian l-cycle.

As mentioned before, for $l = 0$, a Hamiltonian l-cycle in a k-graph H becomes a perfect matching in H. Moreover, any Hamiltonian $(k-1)$-cycle contains a matching of size $\lfloor n/k \rfloor$. Hence, not surprisingly, the results for Hamiltonian cycles and perfect (or almost perfect) matchings are related.

To our knowledge, the first result relating the minimum degree and the existence of a large (though, far from perfect) matching in a k-graph was obtained by Bollobás, Daykin, and Erdős in [4]. It was further extended to perfect matchings by Daykin and Häggkvist in [7].

Definition 1.3. Let d, k, r, and n satisfy $1 \le d \le k-1$ and k divide $n-r$. We define $m_d^r(k,n)$ to be the smallest integer m such that every n-vertex k-graph H satisfying $\delta_d(H) \ge m$ contains a matching M with $\big|V(M)\big| = n-r$.

In Sections 2 and 3, respectively, we summarize what we know about the parameters $h_d^l(k,n)$ and $m_r^r(k,n)$. We present both, asymptotic and exact results, some with sketches of proofs, as well as pose several open questions. We also discuss the k-partite case and some other related topics.

Throughout the paper we will be giving a particular interest to the cases when $d = k-1$, $l = k-1$, and/or $r = 0$. We will be then suppressing the subscript or the superscript, or both, respectively. For instance, $m_d(k,n) = h_d^0(k,n)$ will stand for the smallest integer m such that every k-graph on n vertices with n divisible by k and $\delta_d \geq m$ contains a perfect matching. For future references we summarize our notation here.

Summary of notation: For n divisible by $k-l$

- $h_d^l(k,n) = \min\big\{ h : \delta_d(H) \geq h \Rightarrow H \text{ contains a Hamiltonian } l\text{-cycle}\big\}$
- $h^l(k,n) = h_{k-1}^l(k,n)$
- $h_d(k,n) = h_d^{k-1}(k,n)$
- $h(k,n) = h_{k-1}^{k-1}(k,n)$,

and for $n-r$ divisible by k

- $m_d^r(k,n) = \min\Big\{ m : \delta_d(H) \geq m \Rightarrow H \text{ contains a matching } M, \big|V(M)\big| = n-r\Big\}$
- $m^r(k,n) = m_{k-1}^r(k,n)$
- $m_d(k,n) = m_d^0(k,n)$
- $m(k,n) = m_{k-1}^0(k,n)$.

The parameters $h_d^l(k,n)$ and $m_d^r(k,n)$ are often referred to as *Dirac-type* thresholds. So far, all known results and conjectures indicate that the Dirac thresholds are asymptotic to $c\binom{n-d}{k-d}$, for some $0 < c < 1$. Therefore, the following observation can be useful.

Remark 1.4. Since, by simple averaging,

$$\delta_{d-1}(H) \geq \frac{n-d+1}{k-d+1} \times \delta_d(H),$$

we have for every $c > 0$ that

$$\delta_d(H) \geq c\binom{n-d}{k-d} \qquad \text{implies} \qquad \delta_{d-1}(H) \geq c\binom{n-(d-1)}{k-(d-1)}.$$

Consequently,

$$h_d^l(k,n) \geq c\binom{n-d}{k-d} \qquad \text{implies} \qquad h_{d-1}^l(k,n) \geq c\binom{n-(d-1)}{k-(d-1)}$$

and

$$h^l_{d-1}(k,n) \le c\binom{n-(d-1)}{k-(d-1)} \qquad \text{implies} \qquad h^l_d(k,n) \le c\binom{n-d}{k-d},$$

and similar implications hold for the parameter $m^r_d(k,n)$ as well.

2. Hamilton Cycles

For most of this section we will deal with the case $d = k-1$ and $l = k-1$ and set $h(k,n) = h^{k-1}_{k-1}(k,n)$ for convenience (see Summary of notation in Section 1). Also for convenience, we will call Hamiltonian $(k-1)$-cycles just Hamiltonian cycles, and k-graphs containing such cycles – Hamiltonian.

In 1952 Dirac [8] proved that $h(2,n) = \lceil n/2 \rceil$. The two following graphs show that this result is tight: the union of two complete graphs $2K_{\lceil n/2 \rceil}$ (with one vertex in common when n is odd) and the complete bipartite graph $K_{\lceil n/2 \rceil - 1, \lfloor n/2 \rfloor + 1}$. The first Dirac-type result for hypergraphs was obtained by Katona and Kierstead who proved in [13] that

$$\left\lfloor \frac{n-k+3}{2} \right\rfloor \le h(k,n) \le \left(1 - \frac{1}{2k}\right) n + O_k(1).$$

As a proof of the lower bound they provided the following construction of an extremal k-graph H_0.

Construction 2.1 ([13])**.** Let $V = V' \cup \{v\}$, $|V| = n \ge k^2 + 1$. Split $V' = X \cup Y$, where, $|X| = \left\lfloor \frac{n-1}{2} \right\rfloor$ and $|Y| = \left\lceil \frac{n-1}{2} \right\rceil$. The edges of H_0 are all k-element subsets S of V such that $|X \cap S| \ne \left\lfloor \frac{k}{2} \right\rfloor$ or $v \in S$. It is shown in [13] that H_0 is not Hamiltonian, while $\delta_{k-1}(H_0) \ge \left\lfloor \frac{n-k+1}{2} \right\rfloor$. Thus,

$$h(k,n) \ge \delta_{k-1}(H_0) + 1 = \left\lfloor \frac{n-k+1}{2} \right\rfloor + 1 = \left\lfloor \frac{n-k+3}{2} \right\rfloor.$$

Katona and Kierstead (implicitly) conjectured that their lower bound is the correct value of $h(k,n)$. Recently, this has been confirmed for $k = 3$, first asymptotically [22], then exactly [27], solving also the corresponding Hamiltonian path problem.

Theorem 2.2 ([27]). *Let H be a 3-graph on n vertices, where n is sufficiently large.*

1. *If $\delta_2(H) \geq \lfloor n/2 \rfloor$ then H has a Hamiltonian cycle. Moreover, for every n there exists a 3-graph H_n such that $\delta(H_n) = \lfloor n/2 \rfloor - 1$ and H_n does not have a Hamiltonian cycle. In other words, $h(3,n) = \left\lfloor \frac{n}{2} \right\rfloor$.*

2. *If $\delta_2(H) \geq \lceil n/2 \rceil - 1$ then H has a Hamiltonian path. Moreover, for every n there exists a 3-graph H_n such that $\delta(H_n) = \lceil n/2 \rceil - 2$ and H_n does not have a Hamiltonian path.*

An analogous question regarding the Dirac threshold for Hamiltonian cycles in k-graphs remains open.

Problem 2.3. *Prove that $h(k,n) = \left\lfloor \frac{n-k+3}{2} \right\rfloor$ for all $k \geq 4$.*

As a step toward solving this problem, it was proved in [24] that $h(k,n) \sim \frac{1}{2}n$, that is, $h(k,n) = \big(1 + o(1)\big)\frac{1}{2}n$, for all $k \geq 3$.

Theorem 2.4 ([24]). *Let $k \geq 3$, $\gamma > 0$, and let H be a k-graph on n-vertices, where n is sufficiently large. If $\delta_{k-1}(H) \geq (1/2+\gamma)n$ edges, then H is Hamiltonian. In other words, $h(k,n) \sim \frac{1}{2}n$.*

A sketch of the proof of Theorem 2.4 from [24] is presented in Section 2.2.

2.1. Dirac thresholds for loose(r) Hamiltonian cycles

For two integers, a and b, let us write $a|b$ if a divides b. As an (almost) immediate consequence of Theorem 2.4 we can asymptotically determine the value $h^l(k,n)$ of the Dirac threshold for Hamiltonian l-cycles for all $1 \leq l \leq k-1$ satisfying the congruence $(k-l) \mid k$.

Corollary 2.5 ([19]). *If $(k-l) \mid k$ and $(k-l) \mid n$, then $h^l(k,n) \sim \frac{1}{2}n$.*

Proof. We will show first that $h^l(k,n) \leq \big(\frac{1}{2} + o(1)\big)n$. Since $(k-l) \mid k$ and $(k-l) \mid n$, every Hamiltonian $(k-1)$-cycle $C_n^{k,k-1}$ contains a Hamiltonian l-cycle $C_n^{k,l}$ (indeed, take every $(k-l)$th edge of $C_n^{k,k-1}$). Thus, we have

$$h^l(k,n) \leq h(k,n) = \left(\frac{1}{2} + o(1)\right) n,$$

where the equation follows from Theorem 2.4.

For the lower bound, assume first that, in addition to $(k-l) \mid k$ and $(k-l) \mid n$, we also have $k|n$. Then, by taking every $\frac{k}{k-l}$th edge of $C_n^{k,l}$, we can find a perfect matching $C_n^{k,0}$ inside $C_n^{k,l}$. Thus, in this case,

$$m(k,n) = h^0(k,n) \le h^l(k,n).$$

By the lower bound (2) given in Section 3 we know that

$$m(k,n) \ge \frac{1}{2}n - k,$$

which completes the proof if $k|n$.

If k does not divide n then still $h^l(k,n) \sim \frac{1}{2}n$ because $h^l(k,n) \ge \frac{1}{2}n - k$ by a simple argument from [19] which uses the following constructions.

Construction 2.6. Let $H_1 = (V,E)$ where $V = A \cup B$, $\frac{1}{2}n - 1 \le |A| \le \frac{1}{2}n + \frac{1}{2}$, $|A|$ is odd, and E consists of all $e \in \binom{V}{k}$ such that $|e \cap V|$ is even. Let $H_2 = (V,E)$ where $V = A \cup B$, $|A| = \lceil \frac{1}{2}n \rceil$, and E consists of all $e \in \binom{V}{k}$ such that $|e \cap V|$ is odd. It is easy to check that $\delta_{k-1}(H_i) \ge n/2 - k$, $i = 1,2$. Moreover, it follows by a parity argument that H_1 contains no Hamiltonian l-cycle if $\frac{k}{k-l}$ is odd, while H_2 contains no Hamiltonian l-cycle if $\frac{k}{k-l}$ is even and $\frac{n}{k-l}$ is odd. The remaining case, when $\frac{k}{k-l}$ and $\frac{n}{k-l}$ are even, can be reduced to one of the two previous cases. ■

In the meantime, the value of $h^l(k,n)$ has been determined asymptotically for all $0 \le l \le k-1$, that is, also when $k-l$ does not divide k. First, Kühn and Osthus proved in [17] that $h^1(3,n) \sim \frac{1}{4}n$ and conjectured that $h^1(k,n) \sim \frac{1}{2(k-1)}n$. This conjecture was proved in [14], and independently in [12], where Hán and Schacht generalized it further, obtaining the asymptotic formula $h^l(k,n) \sim \frac{1}{2(k-l)}n$ for all $1 \le l < \frac{1}{2}k$. In turn, Hán and Schacht conjectured the right result for all values of l which was finally proved by Kühn, Mycroft, and Osthus in [15].

Theorem 2.7 ([15]). *If $k-l$ does not divide k and $(k-l) \mid n$, then*

$$h^l(k,n) \sim \frac{n}{\lceil \frac{k}{k-l} \rceil (k-l)}.$$

(Note that $\lceil \frac{k}{k-l} \rceil = 2$ for $l < k/2$.)

So, the situation is quite peculiar as our next example shows. Let $k = 10$. Then the asymptotic values of $h^l(10, n)$ for $l = 0, 1, 2, \ldots, 9$ are $\frac{1}{2}$, $\frac{1}{18}$, $\frac{1}{16}$, $\frac{1}{14}$, $\frac{1}{12}$, $\frac{1}{2}$, $\frac{1}{12}$, $\frac{1}{12}$, $\frac{1}{2}$, $\frac{1}{2}$.

The lower bound in the above theorem comes from the following construction which sheds some light on the origin of the cumbersome formula.

Construction 2.8. Let $H_3 = (V, E)$ where $V = A \cup B$,

$$|A| = \left\lceil \frac{n}{\lceil \frac{k}{k-l} \rceil (k-l)} \right\rceil - 1 \quad \text{and} \quad E = \left\{ e \in \binom{V}{k} \; : \; |e \cap A| \neq \emptyset \right\}.$$

It follows that $\delta_{k-1}(H_3) = |A|$. Recall that every Hamiltonian l-cycle has $m = n/(k-l)$ edges and maximum degree $\Delta = \lceil \frac{k}{k-l} \rceil$. If there was a Hamiltonian l-cycle in H_3, then A would be its vertex cover. However,

$$|A| \times \Delta = \left(\left\lceil \frac{n}{\lceil \frac{k}{k-l} \rceil (k-l)} \right\rceil - 1 \right) \times \left\lceil \frac{k}{k-l} \right\rceil < n/(k-l),$$

a contradiction.

It seems that it will be very hard to pinpoint the value of $h^l(k, n)$ precisely.

Problem 2.9. *Determine the exact value of $h^l(k, n)$ for all $k \geq 3$, $0 \leq l \leq k-1$ and all (sufficiently large) n.*

So far this has been solved for $k = 3$, $l = 2$ in [27] (see Theorem 2.2 above) and for $k \geq 3$, $l = 0$ in [23] (see Theorem 3.4 in Section 3).

2.2. An outline of the proof of Theorem 2.4

In this section we assume that $\delta_{k-1}(H) \geq (1/2 + \gamma)n$ for $\gamma > 0$ and sufficiently small with respect to k. The proof in [24] is built around the notion of an absorbing path. A *k-uniform (tight) path P of length s* is a k-graph with s vertices and $s - k + 1$ edges whose vertices can be ordered $v_1, \ldots, v_s$ in such a way that every k consecutive vertices form an edge (each path has exactly two such orderings). The sequences $(v_1, \ldots, v_{k-1})$ and $(v_s, \ldots, v_{s-k+1})$ are called *the ends* of P, and we say that P *connects* them.

Lemma 2.10 (Absorbing Lemma, [24]). *There exists a path A in H (called* absorbing*) with $\left|V(A)\right| \le 16k\gamma^{k-1}n$ such that for every subset $U \subset V \setminus V(A)$ of size $|U| \le 2^{k-4}\gamma^{2k}n$ there is a path A_U in H with $V(A_U) = V(A) \cup U$ and such that A_U has the same ends as A.*

In other words, the above lemma asserts that there is *one*, not too long path such that *every* not too large subset of vertices can be "absorbed" into the "interior" of this path.

The idea of the proof of Theorem 2.4 can be described in three steps (see Figure 2).

Outline of proof of Theorem 2.4.

1. Fix an absorbing path A guaranteed by Lemma 2.10.
2. Build a cycle C of length at least $n - 2^{k-4}\gamma^{2k}n$ containing A.
3. Applying the absorbing property of A to the set $U = V(H) \setminus V(C)$, insert U into A, obtaining a Hamiltonian cycle $C_{\mathbf{HAM}}$ in H.

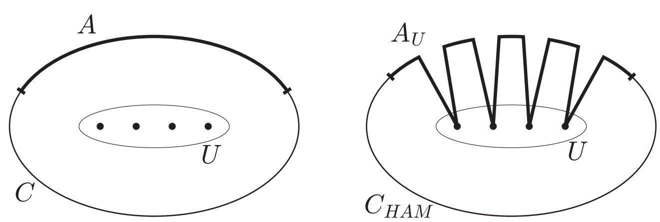

Fig. 2. A bird's view of the proof of Theorem 2.4

Below we explain how these three steps are implemented.

Step 1. The absorbing path will be constructed from absorbing sequences.

Definition 2.11. Given a vertex v, we say that a $(2k-2)$-element sequence of vertices $\mathbf{x} = (x_1, \ldots, x_{2k-2})$ *absorbs* v in H if

- for every $i = 1, \ldots, k-1$ we have $\{x_i, x_{i+1}, \ldots, x_{i+k-1}\} \in H$ (that is, $\mathbf{x}$ spans a path in H) and
- for every $i = 1, \ldots, k$ we also have $\{x_i, x_{i+1}, \ldots, x_{i+k-2}, v\} \in H$ (that is, $\mathbf{x}$ spans a $(k-1)$-uniform path in the link $H(v)$ of v in H).

If $\mathbf{x}$ is actually a segment of a path P and v is not a vertex of P, then P can "absorb" v by replacing the edges $\{x_i, x_{i+1}, \ldots, x_{i+k-1}\}$, $i = 1, \ldots, k-1$, by $\{x_i, x_{i+1}, \ldots, x_{i+k-2}, v\}$, $i = 1, \ldots, k$. This way, the segment $\mathbf{x}$ of P is replaced by the new segment $\mathbf{x}' = (x_1, \ldots, x_{k-1}, v, x_k, \ldots, x_{2k-2})$.

A key feature of absorbing sequences is that there are plenty of them.

Claim 2.12. *For every $v \in V(H)$, there are at least*

$$2^{k-2}\gamma^{k-1}n^{2k-2}$$

sequences absorbing v in H.

Proof. While constructing a v-absorbing sequence $\mathbf{x} = (x_1, \ldots, x_{2k-2})$, there is no restriction on the vertices $x_1, \ldots, x_{k-2}$ other than they should be different from v. Thus, $x_1, \ldots, x_{k-2}$ can be chosen in precisely $(n-1)_{k-2}$ ways. By the degree assumption applied to the set $\{x_1, \ldots, x_{k-2}, v\}$, there are at least $(1/2 + \gamma)n$ vertices x_{k-1} such that $\{x_1, \ldots, x_{k-1}, v\} \in H$.

By the degree assumption applied to the sets $\{x_1, \ldots, x_{k-1}\}$ and $\{x_2, \ldots, x_{k-1}, v\}$, there are at least $2\gamma n + k - 2 > 2\gamma n$ vertices x_k such that

$$\{x_1, \ldots, x_k\} \in H \quad \text{and} \quad \{x_2, \ldots, x_k, v\} \in H.$$

(See Fact 3.1 in [24] for details.) Similarly, for each $i = k+1, \ldots, 2k-2$, there are at least $2\gamma n + k - 2$ vertices x_i such that

$$\{x_{i-k+1}, \ldots, x_i\} \in H \quad \text{and} \quad \{x_{i-k+2}, \ldots, x_i, v\} \in H.$$

Among them, at least $2\gamma n + k - 2 - (i - k) \geq 2\gamma n$ satisfy $x_i \neq x_1, \ldots, x_{i-k}$. Altogether, this implies that there are at least

$$(n-1)_{(k-2)}(1/2 + \gamma)n(2\gamma n)^{k-1} > 2^{k-2}\gamma^{k-1}n^{2k-2}$$

sequences $\mathbf{x} = (x_1, \ldots, x_{2k-2})$ absorbing v. ■

The construction of an absorbing path consists of two phases:

1(a) Selecting a small number of *disjoint,* absorbing sequences such that each vertex is absorbed by many of them;

1(b) Connecting these sequences into one path.

Phase 1(a). We select randomly, with probability $p = \gamma^{k+1}/n^{2k-3}$, a family $\mathcal{R}$ of $(2k-2)$-element sequences $\mathbf{x}$ of vertices. By standard probabilistic argument and by Claim 2.12 it follows that with positive probability $\mathcal{R}$ contains a subfamily $\mathcal{F}$ of at most $2\gamma^{k+1}n$ *disjoint* sequences such that for every vertex v at least $2^{k-4}\gamma^{2k}n$ of these sequences are v-absorbing (see [24] for details).

Phase 1(b). This phase is executed with the help of the connecting lemma from [24], the proof of which is omitted here.

Lemma 2.13 (Connecting Lemma, [24]). *If $\delta_{k-1}(H) \geq (1/2+\gamma)n$ then, for every two disjoint $(k-1)$-element sequences of vertices of H, there is a path in H of length at most $2k/\gamma^2$ which connects them.*

We use Lemma 2.13, but with $\gamma/2$ instead of γ, to connect, one by one, all sequences of $\mathcal{F}$ obtaining an absorbing path. This is possible, because the whole path will have at most

$$|\mathcal{F}| \times (8k/\gamma^2) \leq 2\gamma^{k+1}n \times (8k/\gamma^2) = 16k\gamma^{k-1}n$$

vertices, and thus, at any given time of the connecting procedure, the subhypergraph H^* spanned by the remaining vertices will have

$$\delta_{k-1}(H^*) \geq (1/2+\gamma)n - 16k\gamma^{k-1}n > (1/2+\gamma/2)n > (1/2+\gamma/2)\big|V(H^*)\big|,$$

for sufficiently small $\gamma > 0$.

Step 2. The process of finding a long cycle containing A, can be broken up into three phases:

2(a) Selecting a small “reservoir set” R such that $|R| = 2^{k-5}\gamma^{2k}n$, $R \cap V(A) = \emptyset$, and $H[R]$ inherits the degree property of the entire k-graph H, scaled down to its size.

2(b) Constructing, via The Weak Regularity Lemma, a constant size collection of long, disjoint paths in $H' = H\big[V \setminus \big(V(A) \cup R\big)\big]$, covering all but at most $2^{k-5}\gamma^{2k}\big|V(H')\big|$ vertices of H'.

2(c) Connecting these paths and the absorbing path A into one cycle, utilizing a small chunk of R.

Phase (a) is necessary, since toward the end of the connecting phase (c), there will be only few vertices left outside the path under construction, and thus available for connecting. We make sure, however, that this residual part of H' will contain a small "copy" of H, namely $H[R]$ or its large portion $H[R']$, $R' \subset R$, and so, we will be in position to apply an analog of Lemma 2.13 to it.

Phase 2(a).

Lemma 2.14 (Reservoir Lemma)**.** *There exists a subset $R \subset V \setminus V(A)$ of size $|R| = \lfloor 2^{k-5}\gamma^{2k} n \rfloor$ such that for every $(k-1)$-element set $S \subset V$ we have*

$$\big| N_H(S) \cap R \big| \geq (1/2 + \gamma/2)|R|. \tag{1}$$

Proof. Select R randomly. By Chernoff's bound, with high probability, the set R will satisfy (1). ■

Phase 2(b).

Lemma 2.15 (Path Cover Lemma)**.** *All but at most $2^{k-5}\gamma^{2k}\big|V(H')\big|$ vertices of $H' = H\big[V \setminus \big(V(A) \cup R\big)\big]$ can be covered by at most $m = m(\gamma)$ vertex-disjoint paths $P_1, \ldots, P_m$.*

Proof. See [24]. ■

Phase 2(c). In this final phase of Step 2, we use a lemma which was implicitly proved in [24].

Lemma 2.16 (Restricted connecting Lemma)**.** *Let R be as in Lemma 2.14. Then for every two disjoint, $(k-1)$-element sequences $(x_1, \ldots, x_{k-1})$ and $(y_1, \ldots, y_{k-1})$ of vertices of H, there is a path P in H of length at most $8k/\gamma^2 + 2(k-1)$, which connects them and such that*

$$V(P) \setminus \{x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}\} \subset R.$$

Proof. By property (1) there exist distinct vertices $u_1, \ldots, u_{k-1} \in R$ and $v_1, \ldots, v_{k-1} \in R$ such that $Q_x = (x_1, \ldots, x_{k-1}, u_1, \ldots, u_{k-1})$ and $Q_y = (y_1, \ldots, y_{k-1}, v_1, \ldots, v_{k-1})$ form paths in H. Now, we can apply Lemma 2.13 with $\gamma/2$ to the k-graph $H[R]$ and the sequences $(u_1, \ldots, u_{k-1})$ and $(v_1, \ldots, v_{k-1})$, obtaining a path Q of length $8k/\gamma^2$ connecting them. Then, the path $P = Q_x Q Q_y$ connects $(x_1, \ldots, x_{k-1})$ with $(y_1, \ldots, y_{k-1})$ and has length $\big|V(Q)\big| + 2(k-1)$. ■

Now, we are ready to prove the main lemma of this phase.

Lemma 2.17 (Long Cycle Lemma). *There is in H a cycle C of length at least $n - 2^{k-5}\gamma^{2k}n$ containing A.*

Proof. We perform $m+1$ applications of Lemma 2.16, with $\gamma/3$ instead of $\gamma/2$, to large subsets $R' \subseteq R$, and connect all paths $P_1, \ldots, P_m$, as well as the absorbing path A, into one long cycle C in H. Suppose that at some point we are to connect an end $(x_1, \ldots, x_{k-1})$ of one path with an end $(y_1, \ldots, y_{k-1})$ of another path. Let us denote the yet unused portion of R by R'. Since we connect only $O(1)$ times, each time using only $O(1)$ vertices of R, throughout the procedure we maintain that $|R'| = |R| - O(1) > (1-\gamma/6)|R|$, and thus, by property (1) of R, for every $(k-1)$-element set $S \subset V$ we still have

$$\begin{aligned}\left|N_H(S) \cap R'\right| &\geq (1/2+\gamma/2)|R| - \big(|R| - |R'|\big) > (1/2+\gamma/3)|R| \\ &> (1/2+\gamma/3)|R'|.\end{aligned}$$

Hence, we apply Lemma 2.16 with $\gamma/3$ instead of $\gamma/2$, and so, the obtained connecting paths are of lengths at most $18k/\gamma^2 + 2(k-1)$.

Let T be the set of vertices of H' not covered by the paths $P_1, \ldots, P_m$. Only a subset R' of R and the set T are uncovered by the cycle C. The union of these two sets has size at most $|R| + |T| \leq 2^{k-4}\gamma^{2k}n$ (see Figure 3). ■

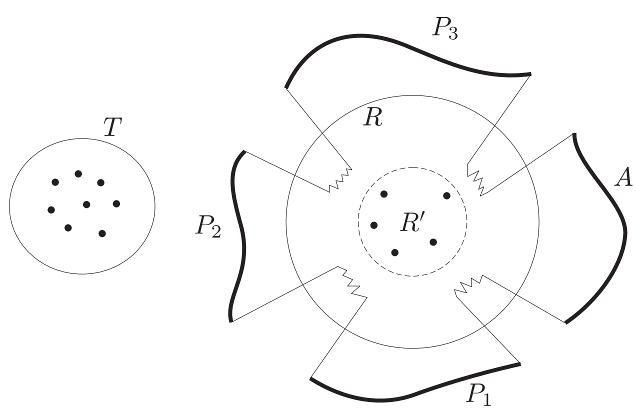

Fig. 3. Phase 2(c) of the proof of Theorem 2.4

Step 3. Let $U = R' \cup T$. Note that $|U| \leq 2^{k-4}\gamma^{2k}n$. Let A_U be the path as defined in Lemma 2.10. Then, replacing A with A_U in C yields a Hamiltonian cycle in H.

This completes the outline of the proof of Theorem 2.4.

2.3. Hamilton cycles in hypergraphs with large vertex minimum degree

There are virtually no results on $h_d^l(k,n)$ for $d \le k-2$. Here we consider the smallest unsolved case: $k = 3$ and $d = 1$.

Two constructions set the bound $h_1(3,n) \ge \left(\frac{5}{9} + o(1)\right)\binom{n-1}{2}$. One is obtained by modifying the hypergraph H_0 from Construction 2.1. We now take $V = X \cup Y$, where $|Y| \sim 2|X|$ (instead of $|Y| \sim |X|$) and all triples S of vertices with $|S \cap X| \ne 1$ as the edges. Let H_0' be the obtained 3-graph. Then

$$\delta_1(H_0') = \max\left(\binom{|Y|-1}{2} + \binom{|X|}{2}, \binom{|X|-1}{2} + \big(|X|-1\big)|Y|\right)$$
$$\sim \frac{5}{9}\binom{n-1}{2}.$$

and, likewise in H_0, there is no Hamiltonian cycle in H_0'.

The other construction is very similar to the hypergraph H_3 described in Construction 2.8. We define H_4 as a hypergraph on the vertex set $V = X \cup Y$, where $|X| = n/3 - 1$, and with the edge set consisting of all triples intersecting X. Then, again, $\delta_1(H_4) \sim \frac{5}{9}\binom{n-1}{2}$ and H_4 has no Hamiltonian cycle.

Note that (for n divisible by 3) the hypergraph H_4 does not even have a perfect matching. As we will see in Section 3 (see Theorem 3.4 below, proved in [11]), the threshold $m_1(3,n)$ for the existence of a perfect matching is, in fact, $\left(\frac{5}{9} + o(1)\right)\binom{n-1}{2}$. Judging by the similarities between Dirac thresholds for perfect matchings and Hamiltonian cycles in various situations, it was tempting to conjecture that $h_1(3,n) \sim m_1(3,n)$. However, even showing that $h_1(3,n) \le c\binom{n-1}{2}$ for some $c < 1$ does not seem to be completely trivial. In our preliminary reconnaissance of this problem, by adapting the original proof from [24] and using Theorem 3.4 along the way, we were able to obtain only the upper bound $h_1(3,n) \le \left(\frac{11}{12} + \gamma\right)\binom{n-1}{2}$. Very recently we learned from Endre that he knows how to prove that, indeed, $h_1(3,n) \sim m_1(3,n)$.

Endre's insight and the existing results showing that $h(k,n) \sim m(k,n)$ for all k suggest that the same is true in general.

Conjecture 2.18. For all $1 \le d \le k-1$,

$$h_d(k,n) \sim m_d(k,n).$$

Note that formula (4) and Conjecture 3.6 in Section 3.2 specify the value of $m_d(k,n)$.

2.4. The k-partite case

Unlike matchings (see the next section) there are very few results on the Hamiltonicity of partite hypergraphs. For graphs, Moon and Moser [20] extended Dirac's theorem to bipartite graphs. Later, the authors of [5] provided a generalization to balanced k-partite graphs. Here we treat briefly the case of k-partite k-graphs, $k \geq 3$.

A k-graph H is k-partite if its vertices can be partitioned into k classes, $V(H) = V_1 \cup \cdots \cup V_k$, in such a way that for every edge $e \in H$ and each $i = 1, \ldots, k$, we have $|e \cap V_i| = 1$. Given such a partition, we call a set of vertices S *legal* if for each $i = 1, \ldots, k$, $|S \cap V_i| \leq 1$. We denote by $\delta'(H) := \delta'_{k-1}(H)$ the minimum of $\deg_H(S)$ taken over all legal $(k-1)$-tuples S in H.

An adaptation of the proof of Theorem 2.4 leads to the following result, which, in turn, implies Theorem 2.4 by taking a random k-partition.

Proposition 2.19. *Let $k \geq 3$, $\gamma > 0$, and let H be a k-partite k-graph on kn vertices with a given equitable partition $V_1, \ldots, V_k$, $|V_i| = n$, where n is sufficiently large. If $\delta'_{k-1}(H) \geq (1/2+\gamma)n$ edges, then H is Hamiltonian. Moreover, there is a k-partite k-graph H_0 on kn vertices and with an equitable partition such that $\delta'_{k-1}(H_0) \geq \left\lfloor \frac{1}{2}n \right\rfloor$ and H_0 does not have a Hamiltonian cycle.*

To obtain H_0, we modify Construction 2.1.

Construction 2.20. Given k and n, let $X = X_1 \cup \cdots \cup X_k$, $Y = Y_1 \cup \cdots \cup Y_k$, and $V_i = X_i \cup Y_i$, $i = 1, \ldots, k$, where all sets X_i and Y_i are pairwise disjoint, $\lfloor kn/2 \rfloor \leq |X|, |Y| \leq \lceil kn/2 \rceil$, $|X| + |Y| = kn$, and, for $i = 1, \ldots, k$, $\lfloor n/2 \rfloor \leq |X_i|, |Y_i| \leq \lceil n/2 \rceil$, and $|V_i| = n$.

Let H_0 be a k-graph with $V = V_1 \cup \cdots \cup V_k = X \cup Y$ whose edge set consists of all k-element subsets S of V such that $|X \cap S| \neq \left\lfloor \frac{k}{2} \right\rfloor$ and $\left| S \cap (V_i) \right| \leq 1$, $i = 1, 2, \ldots, k$. Being a subhypergraph of the k-graph from Construction 2.1, this new H_0 is not Hamiltonian either. Moreover, for every $(k-1)$-element subset S of V, if $|X \cap S| \in \left\{ \left\lfloor \frac{k}{2} \right\rfloor - 1, \left\lfloor \frac{k}{2} \right\rfloor \right\}$, then $\deg_{H_0}(S) \in \{|X_i|, |Y_i|\} = \left\{ \lfloor n/2 \rfloor, \lceil n/2 \rceil \right\}$, while if $|X \cap S| \notin \left\{ \left\lfloor \frac{k}{2} \right\rfloor - 1, \left\lfloor \frac{k}{2} \right\rfloor \right\}$ then $\deg_{H_0}(S) = |V_i| = n$, where i is the unique index such that $S \cap V_i = \emptyset$.

The proof of the main part of Proposition 2.19 follows the lines of the proof of Theorem 2.4 from [24], outlined in Section 2.2. It only needs to be substantially altered in the construction of the absorbing path. Below we provide details of this modified absorbing scheme.

Note that on every path or cycle the cyclical order in which the first edge meets the sets $V_1, \dots, V_k$ is maintained by all subsequent edges. Without loss of generality, we choose $V_1, V_2, \dots, V_k$ as the canonical order, and will be assuming that the absorbing path we build as well as the final Hamiltonian cycle will follow that order.

We will use two different absorbing strategies depending on whether a given set of k vertices which is to be absorbed forms an edge in H or not.

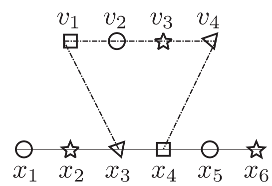

Fig. 4. Absorbing sequence, $k = 4$, the partition sets marked by different symbols

Definition 2.21. For an edge $e = \{v_1, \dots, v_k\} \in H$, where $v_i \in V_i$, $i = 1, \dots, k$, we say that a $(2k-2)$-element sequence of vertices $\mathbf{x} = (x_1, \dots, x_{2k-2})$, *absorbs* e in H if

(a) $x_1 \in V_2, x_2 \in V_3, \dots, x_{k-1} \in V_k, x_k \in V_1, \dots, x_{2k-2} \in V_{k-1}$,

(b) for every $i = 1, \dots, k-1$, we have $\{x_i, x_{i+1}, \dots, x_{i+k-1}\} \in H$ (that is, $\mathbf{x}$ spans a path in H),

(c) for every $i = 1, \dots, k-1$, we have $\{x_i, \dots, x_{k-1}, v_1, \dots, v_i\} \in H$, and

(d) for every $i = 2, \dots, k$, we have $\{v_i, \dots, v_k, x_k, \dots, x_{k-2+i}\} \in H$. (Properties (c) and (d) together imply that the sequence $\mathbf{x}' = (x_1, \dots, x_{k-1}, v_1, \dots, v_k, x_k, \dots, x_{2k-2})$ spans a path in H.)

If $\mathbf{x}$ is actually a segment of a path P and $v_1, v_2, \dots, v_k$ are not on P, then P can "absorb" all these vertices by replacing the segment $\mathbf{x}$ with the new segment $\mathbf{x}'$ (see Fig. 4).

In the final stage of the proof of Proposition 2.19 the above absorbing technique can be used for as long as there are edges induced by the vertices remaining outside the long cycle. When the set of such vertices becomes independent, we use a swapping device which will exchange some k vertices outside the cycle with a set of k vertices which form an edge of H, allowing us to use again the absorbing device and absorb the released vertices back into the cycle.

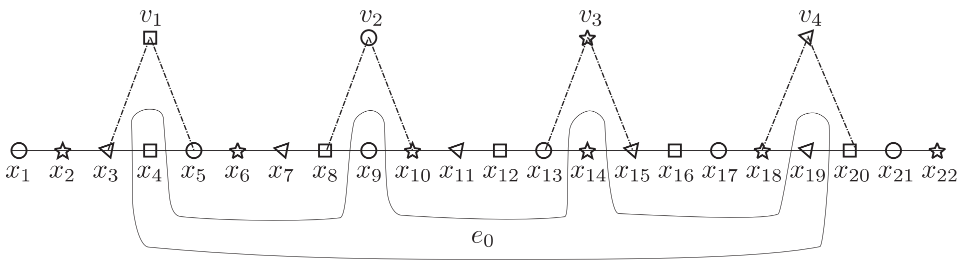

Fig. 5. Swapping sequence, $k = 4$, the partition sets marked by different symbols

Definition 2.22. For a set $S = \{v_1, \ldots, v_k\} \subset V(H)$, where $v_i \in V_i$, $i = 1, \ldots, k$, we say that a $(k^2 + 2k - 2)$-element sequence of vertices $\mathbf{x} = (x_1, \ldots, x_{k^2+2k-2})$, *is edge-swapping for* S if

(a) $x_1 \in V_2, x_2 \in V_3, \ldots, x_{k-1} \in V_k, x_k \in V_1, \ldots, x_{k^2+2k-2} \in V_{k-1}$,

(b) the sequence $\mathbf{x}$ spans a path P_1 in H,

(c) the sequence $\mathbf{x}$ with each x_{ik+i-1} replaced by v_i, $i = 1, \ldots, k$, spans a path P_2 in H, and

(d) $e_0 := \{x_k, x_{2k+1}, \ldots, x_{k^2+k-1}\} \in H$.

If $\mathbf{x}$ is actually a segment of a path P and $v_1, v_2, \ldots, v_k$ are not on P, then P can "swap" the vertices $x_k, x_{2k+1}, x_{3k+2} \ldots, x_{k^2+k-1}$ for $v_1, v_2, \ldots, v_k$ by replacing P_1 with P_2, and thus, releasing the vertices of e_0 from P (see Fig. 5).

So, our absorbing strategy is as follows: create two, disjoint, not too long paths: an absorbing path A containing many absorbing sequences for each *edge* of H, and a swapping path B containing many edge-swapping sequences for each k-element set of vertices of H.

To successfully complete this task all we need are two statements analogous to Claim 2.12. Let us begin with counting, for a given edge $\{v_1, \ldots, v_k\} \in H$, the number of absorbing sequences.

Claim 2.23. *For every edge* $\{v_1, \ldots, v_k\} \in H$, *there are at least* $\gamma^{k-1} n^{2k-2}$ *absorbing sequences in* H.

Proof. As for each $i = k-1, k-2, \dots, 1$, $\deg_H(x_{i+1}, \dots, x_{k-1}, v_1, \dots, v_i) \geq (1/2+\gamma)n$, there are at least $(n/2)^{k-1}$ choices of $x_{k-1}, \dots, x_1$, selected in that order. Then, each of $x_k, \dots, x_{2k-2}$ must be a common neighbor of two $(k-1)$-tuples of already existing vertices, and so there are at least, roughly, $(2\gamma n)^{k-1}$ choices of these vertices. Altogether, we have at least $\gamma^{k-1}n^{2k-2}$ such sequences. ■

Claim 2.24. *For every set $S = \{v_1, \dots, v_k\} \subset V(H)$, there are at least $2^{k^2-k}\gamma^{k^2}n^{k^2+2k-2}$ edge-swapping sequences in H.*

Proof. For a given set $S = \{v_1, \dots, v_k\} \subset V(H)$, we will proceed systematically and count, for each $i = 1, \dots, k^2+2k-2$, the number of choices of x_i, given that $x_1, \dots, x_{i-1}$ have been already selected. There are, roughly, n choices for each of $x_1, \dots, x_{k-2}$ as there are no constraints on them. The vertex x_{k-1} must be a neighbor of $\{x_1, \dots, x_{k-2}, v_1\}$ and then, x_k must be a neighbor of $\{x_1, \dots, x_{k-1}\}$, yielding at least $n/2$ choices of each. The vertices $x_{k+1}, \dots, x_{2k-1}$ are each a common neighbor of two $(k-1)$-tuples of already existing vertices, one on the path P_1, the other on P_2. This is also true for x_{2k}, although for a different reason. Indeed, the paths P_1 and P_2 run together between x_{k+1} and x_{2k}, however x_{2k} must be a common neighbor of $\{x_{k+1}, \dots, x_{2k-1}\}$ and $\{x_{k+2}, \dots, x_{2k-1}, v_2\}$. Then, x_{2k+1} has to be a neighbor of only one $(k-1)$-tuple (the one on P_1, namely $\{x_{k+1}, \dots, x_{2k}\}$). This pattern continues for the next $k-2$ intervals of length $k+1$, until we reach x_{k^2+k-1} which, in addition, has to be a neighbor of $x_k, x_{2k+1}, \dots, x_{k^2-2}$. It is crucial for the success of our construction that no vertex needs to be a common neighbor of three or more already existing $(k-1)$-tuples.

Hence, altogether, there are at least

$$n^{k-2} \times (n/2)^k \times (2\gamma n)^{k^2} = 2^{k^2-k}\gamma^{k^2}n^{k^2+2k-2}$$

choices of the entire edge-swapping sequence. ■

The rest of the proof of a k-partite version of the absorbing lemma follows mutatis mutandis the proof from [24] described in Section 2.2, except that we need to be careful to maintain the canonical order on all paths we build.

3. Perfect Matchings

There are several results on Dirac-type degree thresholds for perfect matchings in k-graphs. To some extent, they resemble the results for Hamiltonian cycles and are often obtained by methods based on similar ideas, most notably, the idea of absorption. In this section we give an overview of such results.

Recall that the Dirac-type threshold $m_d^r(k,n)$ has been introduced in Defintion 1.3 and that we suppress the subscript d when $d = k-1$ as well as we suppress the superscript r when $r = 0$, that is, when we consider perfect matchings (see the Summary of notation in Section 1).

For graphs, an easy argument shows that $m(2,n) = n/2$. Since, for n divisible by k, every Hamiltonian cycle contains a perfect matching, it follows from [24] that $m(k,n) \le n/2 + o(n)$. In [16], Kühn and Osthus sharpened this bound to $m(k,n) \le n/2 + 3k^2\sqrt{n \log n}$, using a result for the k-partite case which they had shown first (see Subsection 3.4). This was further improved in [23] to $m(k,n) \le n/2 + C \log n$, using the idea of absorption. The authors of [25] found a fairly simple proof of the inequality $m(k,n) \le n/2 + k/4$, based on a beautiful idea of Aharoni, Georgakopoulos, and Sprüssel [1] (see Subsection 3.4).

This last bound is very close to the true value of $m(k,n)$. Indeed, constructions presented in [26] yield the lower bound

(2)
$$m(k,n) \ge t(n,k) := \begin{cases} n/2 + 3 - k & \text{if } k/2 \text{ is even and } n/k \text{ is odd,} \\ n/2 + 5/2 - k & \text{if } k \text{ is odd and } (n-1)/2 \text{ is odd,} \\ n/2 + 3/2 - k & \text{if } k \text{ is odd and } (n-1)/2 \text{ is even,} \\ n/2 + 2 - k & \text{otherwise.} \end{cases}$$

Moreover, the main result of [26] shows that, in fact, there is equality in (2).

Theorem 3.1 ([26])**.** *For all $k \ge 2$, $m(k,n) = t(k,n)$, where $t(k,n)$ is given by* (2).

When comparing with Problem 2.3, we see that the conjectured Dirac threshold for a Hamiltonian cycle and the above threshold for a perfect matching differ only by an additive term of about $k/2$. In fact, we know that they coincide for $k = 2$ and differ by at most one for $k = 3$ (see Theorem 2.2).

To prove Theorem 3.1, two cases are separately considered in [26]. When H is "close" to one of the critical k-graphs yielding the lower bound (2), one can find a perfect matching in H by "brute force" If, on the other hand, H is far from the critical k-graphs, we apply a version of the absorbing technique.

The absorbing configurations used in [23] and [26] (as well as in [11]), although different from each other, follow the same pattern: given a set $S \subset V(H)$, $|S| = k$, a matching M_1 is *S-absorbing* if the vertex set $V(M_1) \cup S$ spans in H a matching M_2 of size $|M_1| + 1$. Consider a matching M and a set S, $S \cap V(M) = \emptyset$. If M contains an S-absorbing matching M_1, then one can absorb S into M by swapping M_1 for M_2.

The idea of the proofs in [23] and in the "far-from-critical" case in [26] is now transparent and similar to the idea described in the Outline of the proof of Theorem 2.4:

- Find a relatively small matching M_A such that for every set $S \subset V(H)$, $|S| = k$, there is an S-absorbing matching in M_A.
- Build a matching M' in $H' = H - V(M_A)$ which leaves only a set S of k vertices unmatched.
- Apply the absorbing procedure to S.

Building the almost perfect matching M' requires itself a version of the absorbing technique which works for as long as there are more than k vertices uncovered. Adding the last edge represent a more significant difficulty. In the next subsection we will see that if we allow even one vertex to be uncovered the threshold drops significantly. A matching of size $n/k - k + 2$ can be, however, constructed by a standard greedy approach.

3.1. Almost perfect matchings

Here we present results about $m^r(k,n)$ for $r > 0$. The following construction yields the lower bound

$$m^r(k,n) \geq \frac{n-r}{k}$$

for all $r > 0$.

Construction 3.2. With $n = r \pmod k$, let A and B be disjoint sets of sizes $|A| = \frac{n-r}{k} - 1$ and $|B| = n - |A|$. Let H_r be a k-graph on $V = A \cup B$ consisting of all k-element subsets S of vertices which intersect A. Then the largest matching of H_r has size at most $|A|$, and thus, it has at most $k|A| < n - r$ vertices.

In [26] we established that

$$m^r(k, n) = \frac{n-r}{k} \tag{3}$$

holds for all $r \geq k(k-2)$. This was shown by a fairly simple argument involving a greedy algorithm.

Also, using a version of the absorption method, with the sets S of size $k+1$, it was proved in [26] that for all $r > 0$

$$\frac{n-r}{k} \leq m^r(k, n) \leq \frac{n}{k} + O(\log n).$$

This result stands in a striking contrast with Theorem 3.1, where the threshold is around $n/2$. Hence, from the Dirac threshold perspective, an almost perfect matching appears much sooner than a prefect one.

Note that for $0 < r < k$ we have $\frac{n-r}{k} = \left\lfloor \frac{n}{k} \right\rfloor$ which is the size of the largest matching one can possibly have if n is not divisible by k. We feel that the $O(\log n)$ term, brought in by the technicalities of the absorption method, should not be there.

Problem 3.3. *Prove (or disprove) that $m^r(k, n) = \left\lfloor \frac{n}{k} \right\rfloor$ for all $0 < r < k$. In particular, is it true that if $n \neq 0 \pmod 3$ and $\delta_2(H) \geq \left\lfloor \frac{n}{3} \right\rfloor$ then there is a matching in H of size $\left\lfloor \frac{n}{3} \right\rfloor$?*

3.2. The parameter $m_d^r(k, n)$ for $1 \leq d \leq k-2$

Pikhurko [21] proved that for all $d \geq k/2$

$$m_d(k, n) \sim \frac{1}{2}\binom{n-d}{k-d}. \tag{4}$$

His proof is in part based on the ideas from [16]. Similarly as in [16] he proved first a related result for k-partite k-graphs (see Subsection 3.4).

Moreover, in view of Remark 1.4 it was sufficient to prove the lower bound in (4) only for $d = k-1$ and the upper bound in (4) only for $d = \lceil k/2 \rceil$.

The case $d < k/2$ seems to be much harder. The constructions yielding (2) together with Construction 3.2 applied with $r = 0$ give the following, general lower bound:

$$m_d(k,n) \geq \left(\max \left\{ \frac{1}{2}, 1 - \left(\frac{k-1}{k} \right)^{k-d} \right\} + o(1) \right) \binom{n-d}{k-d}. \tag{5}$$

As for the upper bound, Hàn, Person, and Schacht [11], by a similar method as in [21], proved first that for all $0 \leq d \leq k-1$,

$$m_d^{k(d-1)}(k,n) \leq \left(\frac{k-d}{k} + o(1) \right) \binom{n-d}{k-d}. \tag{6}$$

That is, if for a k-graph H, $\delta_d(H)$ is at least as large as the R-H-S of (6) then H contains a matching covering all but $k(d-1)$ vertices. Then, combining (6) with the absorption method, they improved (6) in the lower range of d by showing that for $1 \leq d < k/2$

$$m_d(k,n) \leq \left(\frac{k-d}{k} + o(1) \right) \binom{n-d}{k-d}. \tag{7}$$

Note that for $d = k-1$, (6) is asymptotically the same result as (3). For $d = 1$, on the other hand, (6) is asymptotically equivalent to an old result of Daykin and Häggvist [7].

In the same paper [11] the authors improved (7) in the smallest case of $k = 3$, $d = 1$, achieving asymptotically the lower bound (5):

Theorem 3.4 ([11])**.**

$$m_1(3,n) \sim \frac{5}{9} \binom{n-1}{2}$$

A crucial ingredient of the proof in [11] was a strong version of the Absorbing Lemma for matchings, an analog of Lemma 2.10 from Section 2.2.

Lemma 3.5 ([11], Lemma 10)**.** *For all $\gamma > 0$ and integers $k > d > 0$ there is an n_0 such that for all $n > n_0$ the following holds: Suppose that H is a k-graph on n vertices with $\delta_d(H) \geq (1/2 + 2\gamma)\binom{n-d}{k-d}$, then there exists a matching $M := M_{abs}$ in H such that*

(i) $|M| < \gamma^k n/k$, *and*

(ii) *for every set* $W \subset V \setminus V(M)$ *of size at most* $|W| \leq \gamma^{2k} n$ *and divisible by* k *there exists a matching in* H *covering exactly the vertices of* $V(M) \cup W$.

This success prompted Hàn, Person, and Schacht to conjecture that (5) is the correct asymptotics of $m_d(k,n)$.

Conjecture 3.6 ([11])**.** For all $1 \leq d < k/2$,

$$m_d(k,n) \sim \max\left\{\frac{1}{2}, 1 - \left(\frac{k-1}{k}\right)^{k-d}\right\}\binom{n-d}{k-d}.$$

Observe that with $d = 1$ the above coefficient equals $\frac{5}{9}$ for $k = 3$, $\frac{37}{64}$ for $k = 4$, and $\frac{369}{625}$ for $k = 5$. However, for $d = 2$ and $k = 5$ it is $\frac{1}{2}$.

Very recently, Markström and the second author [19] lowered slightly the general bound (7) by using some ideas behind Theorem 3.4. They proved that for all $1 \leq d < k/2$

$$m_d(k,n) \leq \left(\frac{k-d}{k} - \frac{1}{k^{k-d}} + o(1)\right)\binom{n-d}{k-d}. \tag{8}$$

In the smallest unknown case, $k = 4$, inequality (7) yields a bound $m_1(4,n) \leq \left(\frac{48}{64} + o(1)\right)\binom{n-1}{3}$. It follows from (8) that $m_1(4,n) \leq \left(\frac{47}{64} + o(1)\right)\binom{n-1}{3}$. By some tedious case by case analysis the coefficient can be lowered further to $\frac{42}{64}$ (see [19]), still far from the conjectured $\frac{37}{64}$.

3.3. Fractional perfect matching

A relaxation of the notion of a perfect matching can be obtained by allowing the inclusion of fractional edges into a matching. *A fractional perfect matching* in a k-graph $H = (V,E)$ is a function $w : E \to [0,1]$ such that for each $v \in V$ we have $\sum_{e \ni v} w(e) = 1$. It follows that if an n-vertex k-graph has a fractional perfect matching then $\sum_{e \in H} w(e) = \frac{n}{k}$, which justifies the name.

For every $1 \leq d \leq k-1$, let

$$m_d^*(k,n) = \min\big\{m : \delta_d(H) \geq m \implies H \text{ contains a fractional perfect matching}\big\}.$$

It was proved in [23] that $m^*_{k-1}(k,n) \le \lceil n/k \rceil$, so, again, the threshold is much lower than that for perfect matchings. Moreover, Construction 3.2 with $|A| = \lceil n/k \rceil - 1$ provides an n-vertex k-graph with $\delta_{k-1} = \lceil n/k \rceil - 1$ which has no fractional perfect matching. Hence, we have the following result.

Theorem 3.7 ([23]). $m^*_{k-1}(k,n) = \lceil n/k \rceil$.

The proof of Theorem 3.7 utilizes the Farkas Lemma (see, e.g., [6] or [18]) which asserts that a system of equations $\mathbf{yA} = \mathbf{b}, \mathbf{y} \ge \mathbf{0}$, is solvable if and only if the system $\mathbf{Ax} \ge \mathbf{0}, \mathbf{bx} < 0$, is unsolvable.

Let $\mathbf{A} := \mathbf{A_H}$ be the incidency matrix of a hypergraph H with rows representing the edges and columns representing the vertices of H. We applied Farkas' Lemma with this $\mathbf{A}$ and with $\mathbf{b} = \mathbf{1}$ – the vector of length n whose all entries are equal to 1, and showed that, under the assumption $\delta_{k-1}(H) \ge \lceil n/k \rceil$ the system of inequalities $\mathbf{Ax} \ge \mathbf{0}, \mathbf{1x} < 0$, has no solutions. Hence, there is a solution to $\mathbf{yA} = \mathbf{1}, \mathbf{y} \ge \mathbf{0}$, which determines a fractional perfect matching $w(e) = y_e$ for all $e \in H$.

It turns out that fractional matchings can be used to give an alternative proof of Theorem 3.4, and possibly even to settle Conjecture 3.6 in full generality. Indeed, the following relation holds.

Theorem 3.8. *For every $1 \le d \le k-1$ and every $\alpha > 0$*

$$\frac{m_d(k,n)}{\binom{n-d}{k-d}} \le \max\left(\frac{1}{2}, \frac{m^*_d(k,n)}{\binom{n-d}{k-d}}\right) + \alpha$$

for sufficiently large n.

Observe that, trivially, $m^*_d(k,n) \le m_d(k,n)$. Therefore, if $m^*_d(k,n) \ge \frac{1}{2}\binom{n-d}{k-d}$ then

$$m^*_d(k,n) \sim m_d(k,n) \tag{9}$$

The proof of Theorem 3.8 is based on Theorem 1.1 in [9]. An immediate corollary of that result asserts the existence of an almost perfect matching in a k-graph with all degrees almost equal and all pair degrees much smaller than the vertex degrees. (see the Remark after Theorem 1.1 in [9]). Here we formulate this corollary in the following lemma.

Lemma 3.9 ([9]). *For all k, $\varepsilon > 0$ and $a > 3$ there exists $\tau = \tau(\varepsilon)$ and $n_0 = n_0(\tau)$ such that if $n > n_0$ and H is an n-vertex k-graph satisfying*

1. $(1-\tau)D < \deg_H(v) < (1+\tau)D$ *for some D and all $v \in V$, and*
2. $\delta_2(H) < D/(\log n)^a$

then H contains a matching M_{alm} covering all but at most εn vertices.

The second tool is the Strong Absorbing Lemma 3.5 (see previous section).

Sketch of Proof of Theorem 3.8. Assume that there exists a constant $0 < c < 1$ such that $m_d^*(k,n) \sim c\binom{n-d}{k-d}$. This is not a restriction at all, as we know by (5) that $m_d^*(k,n) = \Theta\left(\binom{n-d}{k-d}\right)$. For any $\alpha > 0$ consider an n-vertex k-graph H, n large, with

$$\delta_d(H) > (c+\alpha)\binom{n-d}{k-d}.$$

Set $\gamma = \alpha/2$ and $\varepsilon = (\alpha/2)^{2k}$. The proof consists of four steps.

1. Find an absorbing matching M_{abs} satisfying properties (i) and (ii) of Lemma 3.5. Set $H' = H \backslash V(M_{abs})$. Note that $\delta_d(H') \geq (c+\alpha/2)\binom{n-d}{k-d}$.
2. Select a spanning subhypergraph H'' of H' satisfying the assumptions of Lemma 3.9 with $D = n^{0.2}$, $\tau = o(1)$ any $a > 0$, and $n \geq n_0(a)$.
3. Find an almost perfect matching M_{alm} in H'' by applying Lemma 3.9. Note that $\left|V(M_{alm})\right| \geq (1-\varepsilon)\left|V(H')\right|$ and thus, $\left|V(M_{alm} \cup M_{abs})\right| \geq (1-\varepsilon)n$.
4. Extend $M_{alm} \cup M_{abs}$ to a perfect matching of H by using the absorbing property (ii) of M_{abs} with respect to $W = V(H') \setminus V(M_{alm})$. ∎

In view of relation (9), in order to prove Conjecture 3.6 it is sufficient to show that

$$m_d^*(k,n) \sim \left(1 - \left(\frac{k-1}{k}\right)^{k-d}\right)\binom{n-d}{k-d}.$$

This is work in progress. We have heard from Endre that he knows how to determine $m_1(3,n)$ exactly.

3.4. The k-partite case

Recall from Section 2.4 that $\delta'(H) := \delta'_{k-1}(H)$ is the minimum of $\deg_H(S)$ taken over all *legal* $(k-1)$-tuples of vertices S in a k-partite k-graph H. Throughout this subsection, we asume that the k-partition $V(H) = V_1 \cup \cdots \cup V_k$ satisfies $|V_1| = \cdots = |V_k| = n$.

In [16], Kühn and Osthus showed that if

$$\delta'_{k-1}(H) \geq n/2 + \sqrt{2n \log n}$$

then H has a perfect matching. Improving this result, Aharoni, Georgakopoulos, and Sprüssel obtained in [1] a surprisingly strong result.

Theorem 3.10 ([1]). *If for every $(k-1)$-tuple of vertices $(v_1, \dots, v_{k-1}) \in V_1 \times \cdots \times V_{k-1}$ we have $\deg_H(v_1, \dots, v_{k-1}) > n/2$ and for every $(v_2, \dots, v_k) \in V_2 \times \cdots \times V_k$ we have $\deg_H(v_2, \dots, v_k) \geq n/2$, then H has a perfect matching. Consequently, if $\delta'(H) > n/2$ then H contains a perfect matching.*

There is an example in [16] (see also Example 1 in [1]) of a k-partite k-graph H_0 with k even and $n = 2 \pmod 4$, such that $\delta'_{k-1}(H_0) = n/2$ and H_0 does not have a perfect matching. For all other values of k and n one can provide similar constructions with $\delta'_{k-1}(H_0) \geq n/2 - 1$, leaving open the possibility that the result from [1] can be strengthen even further.

Problem 3.11. *Assume that k is even or $n \neq 2 \pmod 4$. Is it true that if $\delta'_{k-1}(H) \geq n/2$ then H has a perfect matching? If so, is it sufficient to impose this degree bound only on two types of legal $(k-1)$-tuples, similar to Theorem* 3.10?

In [1] several other open problems and conjectures are posed. We just quote two of them here. The first one is related to $m_1(k,n)$ in the non-partite case. Note that $1 - (1 - 1/k)^{k-1} < 1 - 1/e$ and compare with Problem 3.6 above.

Problem 3.12 ([1]). *Is it true that if $\delta'_1(H) \geq (1 - 1/e)n^{k-1}$ then there is a perfect matching in H?*

Another problem from [1] is to prove the following conjecture. For a subset $I \subseteq [k]$ of indices, let us call a subset S of vertices of H an I-tuple if $|S| = |I|$ and $S \cap V_i \neq \emptyset$ if and only if $i \in I$. (Observe that if S is an I-tuple then, in fact, for all $i \in I$, we have $|S \cap V_i| = 1$.)

Conjecture 3.13 ([1]). Let I be a subset of $[k]$. If $\deg'(S) > \frac{1}{2}n^{k-|I|}$ for every I-tuple S, and $\deg'(S) > \frac{1}{2}n^{|I|}$ for every $\big([k]\setminus I\big)$-tuple S, then H has a perfect matching.

This conjecture was asymptotically verified by Pikhurko in [21], while its fractional version was proved in [1] (cf. Section 3.3 for the definition).

For $d < k-1$, there are also Dirac-type results relating δ'_d with perfect and almost perfect matchings. Already in 1981, Daykin and Häggvist proved that

$$\delta'_1(H) \geq \frac{k-1}{k}\big(n^{k-1}-1\big)$$

guarantees a perfect matching. This was extended in [11]: if

$$\delta'_d(H) > \frac{k-d}{k}n^{k-d} + kn^{k-d-1}$$

then H contains a matching covering all but $k(d-1)$ vertices, and so, a perfect matching for $d = 1$.

The other extreme case, $d = k-1$, has been also studied in [16]. It was proved there that if

$$\delta'_{k-1}(H) \geq \lceil n/k \rceil$$

then there is a matching in H covering at least $n-(k-2)$ vertices from each partition class V_i, $i = 1, \ldots, k$. It is, perhaps, interesting to compare this result with the results of Subsection 3.1 and consider the following analogue of Problem 3.3.

Problem 3.14. *Is it true for every k-partite k-graph H that if $\delta'_{k-1}(H) \geq \lceil n/k \rceil$ then H has a matching covering at least $n-1$ vertices from each partition class?*

3.5. Other packings

In this section we briefly discuss F-packings, that is, tilings of a hypergraph with vertex disjoint copies of F. Given two hypergraphs, F and H, an *F-packing in H* is a set of vertex disjoint copies of F in H. An F-packing is *perfect* if it covers all vertices of H. For n divisible by $\big|V(F)\big|$, let $p_d(k,n;F)$ be the smallest integer p such that whenever a k-graph H on n vertices, with n divisible by $\big|V(F)\big|$, satisfies $\delta_d(H) \geq p$ then H contains a perfect F-packing. In particular, when $F = K_k^{(k)}$ is a single edge, then $p_d(k, n; K_k^{(k)}) = m_d(k, n)$ is the Dirac threshold for perfect matchings.

Unlike for graphs, there are very few results about degree conditions guaranteeing perfect F-packings in hypergraphs. Below we present two problems, both assuming that $k = 3$ and $d = 2$.

In [17] the authors study, among other things, packings of copies of a $(3, 1)$-cycle $C_s^{(3,1)}$ on s vertices, s even (see Definition 1.1). In the smallest case of $s = 4$, that is, the unique 3-graph with 4 vertices and 2 edges, they show that

$$p_2(3, n; C_4^{(3,1)}) \sim n/4. \tag{10}$$

This seems surprising, since the obtained threshold is about twice smaller than the threshold for perfect matchings. For $s \geq 6$, the value of $p_2\big(3, n; C_s^{(3,1)}\big)$ remains unknown, except for large s when $p_2\big(3, n; C_s^{(3,1)}\big) \sim n/4$, but unlike in (10), here the asymptotics is also as $s \to \infty$ (see Theorem 1.2 in [17]).

As for the lower bound, a construction provided in [17] yields that

$$p_2\big(3, n; C_s^{(3,1)}\big) \geq \frac{\lceil s/4 \rceil}{s} n.$$

This is quite interesting, since it shows that for a fixed s not divisible by 4, the threshold constant is strictly larger than $\frac{1}{4}$ (e.g., it is at least $\frac{1}{3}$ for $s = 6$).

Problem 3.15. *Determine* $p_2\big(3, n; C_s^{(3,1)}\big)$, $s \geq 6$, *s even.*

Similar lower bounds are claimed in [17] for $k > 3$ with $\frac{1}{4}$ replaced by $\frac{1}{2(k-1)}$.

In [21], Pikhurko investigated a challenging problem of determinig $p_2\big(3, n; K_4^{(3)}\big)$, where $K_4^{(3)}$ is the complete 3-graph on 4 vertices, and obtained bounds

$$\frac{3}{4} n - 2 \leq p_2\big(3, n; K_4^{(3)}\big) \leq \frac{2 + \sqrt{10}}{6} n + O\big(\sqrt{n \log n}\,\big),$$

where the upper bound was also proved, independently, by Keevash and Sudakov (unpublished). There is some indication that the truth may lie at the lower end. Indeed, another result from [21] states that for $n \geq 15$, if

$$\delta_2(H) \geq \frac{3}{4} n - \frac{5}{4},$$

then there is a $K_4^{(3)}$-packing in H covering all but at most 14 vertices. However, one should remember that divisibility has a big impact on the Dirac thresholds for (almost) perfect matchings; compare, for instance, the values of $m(3,n)$ and $m^1(3,n)$.

Problem 3.16. *Determine* $p_2\big(3,n;K_4^{(3)}\big)$.

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