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On multicolor Ramsey numbers for loose k-paths of length three

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ABSTRACT

We show that there exists an absolute constant A such that for each $k \geq 2$ and every coloring of the edges of the complete k-uniform hypergraph on Ar vertices with r colors, $r \geq r_k$, one of the color classes contains a loose path of length three.

1. Introduction

For $k \geq 2$, a *k-uniform* hypergraph (or briefly, a k-graph) is an ordered pair $H = (V, E)$, where $V = V(H)$ is a finite set and $E = E(H)$ is a subset of the set $\binom{V}{k}$ of all k-element subsets of V. If $E = \binom{V}{k}$, we call H *complete* and denote it by $K_n^{(k)}$, where $n = |V|$. The elements of V and E are called, respectively, the vertices and edges of H. We often identify H with $E(H)$, writing, for instance, $|H|$ instead of $|E(H)|$. The degree of a vertex v in H, $\deg_H(v)$, equals the number of edges of H which contain v. *A star* is a k-graph S with a vertex v contained in all the edges of S. A star is full if it consists of all sets in $\binom{V}{k}$ containing v, that is, if $\deg_S(v) = \binom{n-1}{k-1}$.

For positive integers k and ℓ, a k-uniform hypergraph is called a *loose path of length ℓ*, and denoted further by $P_\ell^{(k)}$, if its vertex set is $\{v_1, v_2, \ldots, v_{\ell(k-1)+1}\}$ and the edge set is

$$\{e_i = \{v_{(i-1)(k-1)+q} : 1 \leq q \leq k\}, \quad i = 1, \ldots, \ell\},$$

that is, each pair of consecutive edges intersects on a single vertex, while all other pairs of edges are disjoint. Let H be a k-uniform hypergraph and $r \geq 2$ be an integer. The *multicolor Ramsey number* $R(H; r)$ is the minimum integer n such that every r-edge-coloring of the edges of $K_n^{(k)}$ yields a monochromatic copy of H.

In this paper, we study the multicolor Ramsey number $R(P_3^{(k)}; r)$ for $P_3^{(k)}$ and r colors. In the graph case, i.e. when $k = 2$, we have $R(P_3^{(2)}; r) = 2r + c_r$, where $c_r \in \{0, 1, 2\}$ and depends on the

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divisibility of r by three (see [16] for credits to various authors). For hypergraphs ($k \geq 3$), even the case of the shorter path $P_2^{(k)}$ is nontrivial and it was only shown in [2] that $R(P_2^{(3)}; r) \sim \sqrt{6r}$ (an exact value was determined for an infinite sequence of integers r). In the same paper, it was shown that $R(L_2^{(3)}; r) = r + c_r$, where $L_2^{(k)}$ is a pair of k-uniform edges sharing exactly two vertices and $c_r \in \{1, 2, 3\}$ depending on the divisibility of r by six. For the k-graph $M_2^{(k)}$ consisting of a pair of disjoint edges, there is the celebrated result of Lovász in [9] yielding that $R(M_2^{(k)}; r) = r + 2k - 1$ which was generalized in [1] to $R(M_\ell^{(k)}; r) = (r-1)(\ell-1) + \ell k$, where $M_\ell^{(k)}$ is a k-uniform matching with ℓ edges.

For other (than matchings) hypergraphs with more than two edges, the only known results for an arbitrary number of colors appear in [6] and [2]. From our perspective, the most interesting among them deals with the *tight* path of length three, $\{abc, bcd, cde\}$, for which the Ramsey number is shown in [2] to be asymptotic to $2r$.

We now return to the *loose* path of length three which, recall, has $3k-2$ vertices. Coloring the k-element subsets of $\{1, \ldots, r+3k-4\}$ by their smallest element i if $i < r$, and otherwise by color r (e.g., see [7], Proposition 3.1) shows that

$$R(P_3^{(k)}; r) \geq r + 3k - 3. \tag{1}$$

It is conjectured that for each $k \geq 3$ and all r there is equality in (1). So far, it has been verified only for $k = 3$ and $r = 2, 3, \ldots, 10$ [5,7,8,15,14]. In fact, for $k = 3$ and $r = 2$ the Ramsey number has been determined for paths of all lengths (see [12] and [13]).

A general upper bound on $R(P_3^{(k)}; r)$, $k \geq 3$, follows by a standard application of Turán numbers. Indeed, it was proved by Füredi, Jiang, and Seiver [4] that for $n \geq n_0(k)$ the unique largest $P_3^{(k)}$-free k-graph on n vertices is the full star (see Lemma 3). From this, it follows that for r large enough

$$R(P_3^{(k)}; r) \leq kr, \tag{2}$$

valid for all $k \geq 3$ and $r \geq r_0(k)$ (see [7], Proposition 3.2). For $k = 3$, it was improved by Łuczak and Polcyn first to $R(P^{(3)}; r) \leq 2r + O(\sqrt{r})$ [11] and, recently, to $R(P^{(3)}; r) \leq 1.98r + 7\sqrt{r}$ [10]. The main goal of this paper is to show that for r large enough $R(P_3^{(k)}; r)/r$ is bounded from above by a constant which does not depend on k.

Theorem 1. *For each $k \geq 3$ there exists r_k such that for all $r \geq r_k$*

$$R(P_3^{(k)}; r) \leq 250r.$$

In view of inequality (1), r_k is at least linear in k, but we have not put any effort into optimizing it. Likewise, constant 250 is probably very far from the optimal value.

2. Proof of Theorem 1

In view of (2), we may assume $r_k \geq r_0(k)$ and restrict ourselves to $k \geq 250$. Our proof uses two results on Turán numbers for loose k-paths of length two and three. The first of them was proved by Frankl in [3].

Lemma 2. *Let $k \geq 4$ and H be a k-uniform hypergraph on n vertices in which no two edges intersect on a single vertex. Then, for large n, $|H| \leq \binom{n-2}{k-2}$.*

The second result, due to Füredi, Jiang, and Seiver [4], deals with the main object of our study, $P_3^{(k)}$, the loose k-uniform path of length three.

Lemma 3. *Let $k \geq 3$ and H be a $P_3^{(k)}$-free k-uniform hypergraph on n vertices. Then, for large n, $|H| \leq \binom{n-1}{k-1}$. Moreover, the unique k-graph H which achieves this bound is the full star S.*

Theorem 1 is a direct consequence of the following 'stability' version of Lemma 3 which states, roughly, that the structure of each $P_3^{(k)}$-free dense k-graph is dominated by a giant star.

Lemma 4. *For every $k \geq 250$ and $n \geq n_0(k)$, each $P_3^{(k)}$-free k-uniform hypergraph H, has a vertex v with degree*

$$\deg_H(v) \geq |H| - 0.96^k \binom{n-1}{k-1}.$$

We defer the proof of Lemma 4 to the next section. Here we show how Theorem 1 follows from it.

Proof of Theorem 1. For a given $k \geq 250$ and $A = 250$, let $r \geq r_k$, where $r_k \geq r_0(k)$ is chosen so that also $250r_k \geq n_0(k)$ with $n_0(k)$ defined in Lemma 4. Suppose that the complete k-graph $K := K_{Ar}^{(k)}$ on Ar vertices is colored with colors $1, 2, \ldots, r$ in such a way that no monochromatic $P_3^{(k)}$ emerges. For every color c choose (possibly with repetitions) a vertex v_c with maximum degree in this color and let $R = \{v_c : c = 1, 2, \ldots, r\}$.

Consider now the complete k-graph H obtained from K by removing all vertices in R. We have $|V(H)| \geq Ar - |R| \geq (A-1)r$ and thus $|H| \geq \binom{(A-1)r}{k}$. On the other hand, by applying Lemma 4 to each color class, we have $|H| \leq r(0.96)^k \binom{Ar-1}{k-1}$. However, since $k \geq A = 250$, we have

$$r(0.96)^k \binom{Ar-1}{k-1} < \binom{(A-1)r}{k}, \tag{3}$$

a contradiction. To see (3), observe first that the two sides of (3) are asymptotic (as r is growing) to, respectively, $0.96^k A^{k-1} r^k/(k-1)!$ and $(A-1)^k r^k/k!$. Thus it remains to show that $(A-1)^k > k(0.96)^k A^{k-1}$, or, equivalently, $(A-1)(1-1/A)^{k-1} > k(0.96)^k$. Now it is enough to observe that $1 - 1/A \geq 0.99$ for $A \geq 250$ and $k < (99/96)^k$ for $k \geq 167$. □

3. Proof of Lemma 4

Let us start with the following two elementary observations.

Fact 5. *Every hypergraph H contains a sub-hypergraph G with minimum degree greater than $\frac{|E(H)|}{|V(H)|}$.*

Proof. Define G as a subhypergraph of H which maximizes the ratio $\frac{|E(G)|}{|V(G)|}$ and has the smallest number of vertices. If for some $v \in V(G)$, $\deg_G(v) \leq \frac{|E(H)|}{|V(H)|}$, then

$$\frac{|E(G-v)|}{|V(G-v)|} \geq \frac{|E(G)| - |E(H)|/|V(H)|}{|V(G)| - 1} \geq \frac{|E(G)|}{|V(G)|},$$

which contradicts our choice of G. □

Fact 6. *Every bipartite graph B with vertex classes V_1 and V_2 contains a subgraph G with $\deg_G(v) \geq |B|/(2|V_i|)$ for every vertex $v \in V(G) \cap V_i$, $i = 1, 2$.*

Proof. Let us remove one by one the vertices with (current) degree smaller than the above bounds. Then, by the time the degrees of all remaining vertices satisfy the required bounds, we remove fewer than

$$|V_1| \times |B|/(2|V_1|) + |V_2| \times |B|/(2|V_2|) = |B|$$

edges, and so the final subgraph G is non-empty. □

Lemma 4 is a straightforward consequence of the following two propositions.

Proposition 7. *For all $k \geq 3$, $b > 0$, and sufficiently large n, the following holds. Let H be a $P_3^{(k)}$-free, n-vertex, k-uniform hypergraph and let, for some $v \in V(H)$, $\deg_H(v) \geq b\binom{n-1}{k-1}$. Then,*

$$\deg_H(v) \geq |H| - \left(1 - \left(\frac{b}{k-1}\right)^{1/(k-2)}\right)^{k-1} \binom{n-1}{k-1}.$$

Proof. Let $H(v)$ be the link of v in H, that is, the $(k-1)$-uniform, $(n-1)$-vertex hypergraph consisting of all $(k-1)$-element subsets of $V(H)$ which together with v form edges in H. Note that $|H(v)| = \deg_H(v)$. Fact 5 implies that there is a subgraph F of $H(v)$ with minimum degree

$$\delta(F) \geq \delta := \frac{b}{k-1}\binom{n-2}{k-2}.$$

Claim 8. *The number of vertices $|V(F)|$ of F is bounded from below by*

$$|V(F)| \geq \left(\frac{b}{k-1}\right)^{1/(k-2)}(n-1).$$

Proof. Since

$$\binom{|V(F)|}{k-1} \geq |F| \geq |V(F)|\frac{b}{(k-1)^2}\binom{n-2}{k-2},$$

it follows that

$$\binom{|V(F)|-1}{k-2} \geq \frac{b}{k-1}\binom{n-2}{k-2},$$

so, using the shorthand notation $(x)_t = x(x-1)\cdots(x-t+1)$,

$$1 \geq \frac{b}{k-1}\frac{(n-2)_{k-2}}{(|V(F)|-1)_{k-2}} > \frac{b}{k-1}\left(\frac{n-1}{|V(F)|}\right)^{k-2},$$

which implies the required bound for $|V(F)|$. □

Claim 9. *Let n be sufficiently large. For every edge $e \in H$, either $v \in e$ or $e \cap V(F) = \emptyset$.*

Proof. Suppose there exists an edge $e \in H$ such that $v \notin e$ and $e \cap V(F) \neq \emptyset$. Let $w \in e \cap V(F)$. Since $\deg_F(w) \geq \delta = \Omega(n^{k-2})$ while the number of edges of F intersecting e on at least two vertices is $O(n^{k-3})$, there is an edge $f' \in F$ such that $e \cap f' = \{w\}$. Further, since $\deg_H(v) \geq b\binom{n-1}{k-1}$ while the number of edges of H containing v and intersecting $e \cup f'$ is $O(n^{k-2})$, there is an edge $h \in H$ such that $v \in h$ and $h \cap (e \cup f') = \emptyset$. The edges $e, f' \cup \{v\}$, and h form a copy of $P_3^{(k)}$ in H, a contradiction. □

In view of Claim 9, to complete the proof of Proposition 7, we bound from above the number of edges of H which do not contain v by $|H - (V(F) \cup \{v\})|$, where $H - (V(F) \cup \{v\})$ is the induced subhypergraph of H obtained by deleting vertex v and all vertices of F. Since H, and thus $H - (V(F) \cup \{v\})$, is $P_3^{(k)}$-free, we can bound $|H - (V(F) \cup \{v\})|$ by the Turán number for $P_3^{(k)}$ given in Lemma 3. Using the bound for $|V(F)|$ given by Claim 8, we thus get

$$\begin{aligned}|H - (V(F) \cup \{v\})| &\leq \binom{n-|V(F)|-2}{k-1}\\ &\leq \binom{n-(n-1)(b/(k-1))^{1/(k-2)}-2}{k-1}\\ &< \binom{(n-1)(1-(b/(k-1))^{1/(k-2)})}{k-1}\\ &\leq \left(1-\left(\frac{b}{k-1}\right)^{1/(k-2)}\right)^{k-1}\binom{n-1}{k-1}.\end{aligned}$$

As $|H| = \deg_H(v) + |H - (V(F) \cup \{v\})|$, this completes the proof of Proposition 7. □

Proposition 10. *For all $k \geq 250$ and sufficiently large n the following holds. If H is a $P_3^{(k)}$-free k-graph on n vertices and $|H| \geq 0.96^k\binom{n-1}{k-1}$, then $\Delta(H) \geq 0.9^k\binom{n-1}{k-1}$.*

Proof. Let H be a $P_3^{(k)}$-free k-graph on n vertices and with $|H| \geq 0.96^k\binom{n-1}{k-1}$. By F we denote the shadow of H, i.e.

$$F = \{f \in [n]^{k-1} : f \subset e \text{ for some } e \in H\}.$$

Let us now suppose that $\Delta(H) < 0.9^k\binom{n-1}{k-1}$. We shall show that this assumption leads to a contradiction.

The main idea of the argument goes roughly as follows. First, we deal with the case when F is small (Claim 11). Then there are many vertices v with large links. Consequently, it is enough to find in F a loose $(k-1)$-path of length two, say f_1, f_2 (and for that, due to Lemma 2 we only require that $|F| = \Omega(n^{k-3})$) and find another f_3 in F so that $(f_1 \cup f_2) \cap f_3 = \emptyset$. Then, for some $v_1, v_2 \in V(H)$, the edges $\{v_1\} \cup f_1$, $\{v_2\} \cup f_2$, and $\{v_2\} \cup f_3$ form a $P_3^{(k)}$ in H.

In the case when F is large we select three large, disjoint subsets of vertices, W_1, W_2 and W_3, and three large, disjoint subsets of F, S_1, S_2, and S_3, such that for each $i = 1, 2, 3$, and every $f \in S_i$, there is a vertex $v \in W_i$ with $f \cup \{v\} \in H$ and, moreover, $f \cap (W_1 \cup W_2 \cup W_3) = \emptyset$. The sets S_i are so large that we are able to find a copy of $P_3^{(k-1)}$ consisting of some sets $f_i \in S_i$, $i = 1, 2, 3$. This path, in turn, can be easily extended to a copy of $P_3^{(k)}$ by enlarging each f_i to $f_i \cup \{v_i\} \in H$, where $v_i \in W_i$.

In order to make the above outline precise, let us start with the following observation.

Claim 11. $|F| \geq \frac{1}{4}|H|$.

Proof. Let us consider an auxiliary bipartite graph B, with vertex classes $V(H)$ and F, and with edge set

$$\{\{v, f\} : \{v\} \cup f \in H\}.$$

Clearly, $|B| = k|H|$. Further, define

$$F' = \{f \in F : |\{e \in H : f \subset e\}| \geq 2k\}$$

and observe that $|F'| \leq \binom{n-1}{k-2}$. Indeed, otherwise, by the Turán number for $P_3^{(k-1)}$, F' would contain a copy of $P_3^{(k-1)}$ which could be easily extended to a copy of $P_3^{(k)}$ in H.

Let B' be the subgraph of B consisting of all edges with one endpoint in F'. We have

$$|B| = \sum_f \deg_B(f) \leq |B'| + (|F| - |F'|)2k,$$

so

$$|F| \geq |F| - |F'| \geq \frac{1}{2k}(|B| - |B'|).$$

Thus, recalling that $|B| = k|H|$, to complete the proof of Claim 11, it suffices to show that

$$|B'| \leq |B|/2. \tag{4}$$

Suppose that $|B'| \geq |B|/2$. Then,

$$|B'| \geq \frac{k}{2}|H| \geq \frac{k}{2}(0.96)^k\binom{n-1}{k-1}.$$

We apply Fact 6 to B', obtaining a subgraph B'' with vertex sets $V_1 \subset V(H)$ and $F'' \subset F'$ such that, for $n \geq k(k-1)$, each vertex $v \in V_1$ has in B'' degree at least

$$\frac{k(0.96)^k\binom{n-1}{k-1}}{4n} \geq \frac{1}{4}(0.96)^k\binom{n-1}{k-2}$$

and each $f \in F''$ has in B'' degree at least

$$\frac{k(0.96)^k\binom{n-1}{k-1}}{4\binom{n-1}{k-2}} \geq \frac{1}{4}(0.96)^k n.$$

Since, for large n, $|F''| \geq \frac{1}{4}(0.96)^k\binom{n-1}{k-2} > \binom{n-2}{k-3}$, by Lemma 2, F'' contains two $(k-1)$-sets f_1, f_2 such that $|f_1 \cap f_2| = 1$. Let N_i be the neighborhood of f_i in B'', $i = 1, 2$. If there was an edge $(v, f) \in B''$ with $v \in N_1 \cup N_2$ (say, $v \in N_2$) and $(f_1 \cup f_2) \cap f = \emptyset$, then the k-sets $\{v_1\} \cup f_1$, $\{v\} \cup f_2$, and $\{v\} \cup f$, where $v_1 \in N_1$, $v_1 \neq v$, would form a copy of $P_3^{(k)}$ in H, a contradiction. Thus, in B'', all neighbors f of vertices in $N_1 \cup N_2$ must intersect $f_1 \cup f_2$. Since $|N_1 \cup N_2| \geq |N_1| \geq \frac{1}{4}(0.96)^k n$, the number of edges of B'' leaving $N_1 \cup N_2$ is at least

$$\frac{1}{4}(0.96)^k n \times \frac{1}{4}0.96^k\binom{n-1}{k-2} = \frac{1}{16}(0.96)^{2k} n\binom{n-1}{k-2}.$$

Each of these edges of B'' represents an edge of H which intersects $f_1 \cup f_2$, a set of size smaller than $2k$. Hence, by averaging, there exists a vertex in $f_1 \cup f_2$ belonging to at least

$$\frac{1}{32k}(0.96)^{2k} n\binom{n-1}{k-2} > 0.9^k\binom{n-1}{k-1}$$

of these edges (note that the last inequality is valid for $k \geq 250$). This contradicts our assumption on $\Delta(H)$ and, therefore, completes the proof of Claim 11. □

To continue with the proof of Proposition 10, for every $f \in F$ we choose just one vertex v_f such that $\{v_f\} \cup f \in H$. Observe that by our assumption on $\Delta(H)$, for each $v \in V(H)$,

$$|\{f \in F : \ v = v_f\}| < 0.9^k\binom{n-1}{k-1}. \tag{5}$$

Further, we split the vertex set $V(H)$ randomly into two parts, U_1 and U_2, where each vertex belongs to U_1 independently with probability $1/k$. We call a set $f \in F$ *proper* if $v_f \in U_1$ and $f \subseteq U_2$.

Let X count the number of proper sets. Since

$$P(f \text{ is proper}) = \frac{1}{k} \cdot \left(\frac{k-1}{k}\right)^{k-1} \geq \frac{1}{k} \cdot \frac{1}{e} > \frac{1}{3k},$$

by Claim 11,

$$EX = \sum_{f \in F} P(f \text{ is proper}) = |F| \cdot P(f \text{ is proper}) \geq \frac{0.96^k}{12k}\binom{n-1}{k-1}.$$

Thus, there exists a partition (U_1, U_2) such that the number of proper sets f is at least $\frac{0.96^k}{12k}\binom{n-1}{k-1}$. For each $v \in U_1$, set

$$F_v = \{f \in F : \ v = v_f \text{ and } f \subset U_2\} \quad \text{and} \quad \phi_v = \frac{|F_v|}{\binom{n-1}{k-1}}.$$

By the above lower bound on the number of proper sets f, we have $\sum_{v \in U_1} \phi_v > 0.96^k/(12k)$ and, by (5), for each v, we have also $\phi_v < 0.9^k$. We partition the set $\{v \in U_1 : \ F_v \neq \emptyset\}$ into three subsets W_1, W_2, W_3 so that the sums $S_i := \sum_{v \in W_i} \phi_v$, $i = 1, 2, 3$, are as close to each other as possible. This can be done, for instance, by a greedy algorithm which places the vertices one after another into the set with the current minimum total of ϕ_v's. Then, assuming that $S_1 \leq S_2 \leq S_3$, we have

$$S_1 > S_3 - 0.9^k \geq \frac{1}{3}(S_1 + S_2 + S_3) - 0.9^k \geq \frac{1}{4}(S_1 + S_2 + S_3),$$

provided

$$0.9^k < \frac{1}{12} \times \frac{1}{12k}0.96^k = \frac{1}{144k}0.96^k \leq \frac{1}{12}(S_1 + S_2 + S_3),$$

which is valid for $k \geq 250$. Hence, for each $i = 1, 2, 3$,

$$\binom{n-1}{k-1}S_i = \sum_{v \in W_i}|F_v| \geq \frac{0.96^k}{48k}\binom{n-1}{k-1}.$$

The sets W_1, W_2, W_3 generate a corresponding partition of the proper sets f into 'colors' $C_i = \bigcup_{v\in W_j} F_v$. In order to complete the proof of Proposition 10, it suffices to show that such a 3-coloring contains a $(k-1)$-path of length three whose edges are colored with different colors. Such a path can be extended to a copy of $P_3^{(k)}$ in H, yielding a contradiction.

However, all sets W_i are so dense that the existence of such a path is an easy consequence of Fact 5. Indeed, recall that in each color there are at least $0.96^k/(48k)\binom{n-1}{k-1}$ edges. Therefore, by Fact 5, in each color C_i, $i = 1, 2, 3$, viewed as a $(k-1)$-graph, one can find a sub-hypergraph G_i with

$$\delta(G_i) \geq \frac{0.96^k}{48k^2}\binom{n-2}{k-2}.$$

Moreover, $|V(G_i)| \geq 0.9n$, since otherwise for each vertex $v \in V(G_i)$,

$$\deg_{G_i}(v) \leq \binom{0.9n}{k-2} < \frac{0.9^{k-2}n^{k-2}}{(k-2)!} < \frac{0.96^k}{48k^2}\binom{n-2}{k-2} \leq \delta_{G_i},$$

where the penultimate inequality holds for $k \geq 250$. Consequently, the intersection of the vertex sets of these three graphs, $U := V(G_1) \cap V(G_2) \cap V(G_3)$, has size $|U| \geq 0.7n$.

Fix a vertex $v \in U$. Since $\deg_{G_1}(v) \geq 0.96^k/(48k^2)\binom{n-2}{k-2}$ and the number of edges of G_1 with $f \cap U = \{v\}$ is at most $\binom{0.3n}{k-2}$, there exists an edge $f_1 \in G_1$ and a vertex $w \in U$, $w \neq v$, such that $\{v, w\} \subset f_1 \cap U$. Moreover, since the number of edges of G_2 containing v and another vertex of f_1 is $O(n^{k-3})$, we can find $f_2 \in G_2$ such that $f_1 \cap f_2 = \{v\}$. Similarly, there exists $f_3 \in G_3$ such that $f_3 \cap (f_1 \cup f_2) = \{w\}$. Then the edges f_2, f_1, and f_3 form a desired copy of $P^{(k-1)}$ in F. Finally, the edges $\{v_{f_i}\} \cup f_i$, $i = 1, 2, 3$, create a k-path $P_3^{(k)}$ in H, a contradiction. □

Proof of Lemma 4. If $|H| < 0.96^k\binom{n-1}{k-1}$ then the assertion obviously holds. Let us assume that $|H| \geq 0.96^k\binom{n-1}{k-1}$. Then, by Proposition 10, there exists a vertex $v \in V(H)$ with

$$\deg_H(v) \geq 0.9^k\binom{n-1}{k-1}.$$

Therefore, by Proposition 7 with $b = 0.9^k$,

$$\deg_H(v) \geq |H| - \left(1 - \left(\frac{0.9^k}{k-1}\right)^{\frac{1}{k-2}}\right)^{k-1}\binom{n-1}{k-1}.$$

Thus, all we need to verify is that

$$\left(1 - \left(\frac{0.9^k}{k-1}\right)^{\frac{1}{k-2}}\right)^{k-1} < 0.96^k.$$

To this end, observe that

$$0.96^{k/(k-1)} > 0.96^2 > 0.9,$$

while

$$1 - \left(\frac{0.9^k}{k-1}\right)^{\frac{1}{k-2}} < 0.9$$

is equivalent to

$$0.1^{k-2}(k-1) < 0.9^k$$

which holds for $k \geq 3$. □

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