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Constructing sparsest ℓ-hamiltonian saturated k-uniform hypergraphs for a wide range of ℓ

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ABSTRACT

Given $k \geq 3$ and $1 \leq \ell < k$, an (ℓ, k)-cycle is one in which consecutive edges, each of size k, overlap in exactly ℓ vertices. We study the smallest number of edges in k-uniform n-vertex hypergraphs which do not contain hamiltonian (ℓ, k)-cycles, but once a new edge is added, such a cycle is promptly created. It has been conjectured that this number is of order n^ℓ and confirmed for $\ell \in \{1, k/2, k-1\}$, as well as for the upper range $0.8k \leq \ell \leq k-1$. Here we extend the validity of this conjecture to the lower–middle range $(k-1)/3 \leq \ell < (k-1)/2$.

1. Introduction

A k-uniform hypergraph H which we will be calling a k-graph, is a family of k-element subsets (edges) of a vertex set V. Given integers $1 \leq \ell < k$, an (ℓ, k)-*cycle* is a k-graph which, for some s divisible by $k-\ell$, consists of distinct vertices $v_1, \ldots, v_s$ and $s/(k-\ell)$ edges

$$\{v_1, \ldots, v_k\}, \ \{v_{k-\ell+1}, \ldots, v_{2k-\ell}\}, \ \ldots, \ \{v_{s-(k-\ell)+1}, \ldots, v_s, v_1, \ldots, v_\ell\}.$$

An (ℓ, k)-*path* is defined similarly. Note that the number of vertices in an (ℓ, k)-path equals ℓ modulo $k-\ell$.

A k-graph H is ℓ-*hamiltonian saturated* (a.k.a. *maximally non-*ℓ*-hamiltonian*) if it is not ℓ-hamiltonian, but adding any new edge results in creating a hamiltonian (ℓ, k)-cycle.

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We are interested in the smallest possible number of edges, denoted by $\mathrm{sat}(n, k, \ell)$, of an *ℓ-hamiltonian saturated* k-graph on n vertices. For graphs, Clark and Entringer [1] proved that $\mathrm{sat}(n, 2, 1) = \lceil 3n/2 \rceil$ for all $n \geq 52$.

As the problem for hypergraphs, introduced in [2,3], seems to be much harder, we are quite satisfied with results estimating the order of magnitude of $\mathrm{sat}(n, k, \ell)$. Listing the results below, we silently assume that n is divisible by $k - \ell$. It was observed in [4], Prop. 2.1, that for all $k \geq 3$ and $1 \leq \ell \leq k - 1$,

$$\mathrm{sat}(n, k, \ell) = \Omega(n^{\ell}) \tag{1}$$

and conjectured that this lower bound gives the correct order of magnitude.

Conjecture 1. *For all $k \geq 3$ and $1 \leq \ell \leq k - 1$*

$$\mathrm{sat}(n, k, \ell) = \Theta(n^{\ell}). \tag{2}$$

In [4,5] we confirmed this conjecture for $\ell = 1$, $\ell = k/2$, as well as for all $0.8k \leq \ell \leq k - 1$, (see [6] for the case $\ell = k - 1$). In [7] we proved a weaker general upper bound

$$\mathrm{sat}(n, k, \ell) = O\left(n^{\frac{k+\ell}{2}}\right) \tag{3}$$

and improved it for some pairs (k, ℓ) in the range $\ell > k/2$. In this paper, our main result sets another general bound on $\mathrm{sat}(n, k, \ell)$ which improves (3) for every pair (k, ℓ) where $(k-2)/5 < \ell < (k-1)/2$.

Theorem 2. *Let $2 \leq \ell < (k - 1)/2$ and $p = \max\{\ell, k - 2\ell - 1, \lceil k/2 \rceil - \ell\}$. Then*

$$\mathrm{sat}(n, k, \ell) = O\left(n^{p}\right).$$

Note that $p < (k+\ell)/2$ when $k-2\ell-1 < (k+\ell)/2$ which is equivalent to $(k-2)/5 < \ell < \lfloor k/2 \rfloor$. The bound in Theorem 2 is strong enough to confirm Conjecture 1 for a new, wide range of ℓ.

Corollary 3. *If $(k - 1)/3 \leq \ell < (k - 1)/2$, then*

$$\mathrm{sat}(n, k, \ell) = \Theta\left(n^{\ell}\right).$$

In particular, the smallest new cases of (k, ℓ) covered by Corollary 3 include $(6, 2)$ and $(7, 2)$.

Our proof follows the general line of that in [5], where the case $\ell = k/2$ was settled, but with significant alterations. First of all, we had to carefully redefine and recalculate many parameters involved in the proof. An additional technical difficulty was that now we allow also odd values of k. However, the main obstacle, compared with the construction in [5], was due to the gap between two consecutive disjoint edges on an (ℓ, k)-path, caused by considering $\ell < k/2$. To overcome this problem, among others, we had to prove new properties of the crucial function ν (see Section 2.1).

2. Construction

We will prove Theorem 2 by constructing, for any large N divisible by $k - \ell$, an ℓ-hamiltonian saturated k-uniform hypergraph on N vertices and with $\Theta\left(N^{p}\right)$ edges. (From now on we use N, as n is reserved for the order of a graph which plays a crucial role in the construction). In this section, we first define some parameters and then describe our construction. We then present a short proof of Theorem 2, the two ingredients of which, Lemmas 10 and 11, will be proved in the last two sections.

2.1. *The function ν*

In our proofs a pivotal role will be played by (ℓ, k)-paths whose every edge draws at least $k-\ell+1$ vertices from the same fixed, relatively small set, while the remaining vertices come from a much larger set. To handle the maximum length of such paths we introduce the following function.

Definition 4 (*Function* ν)**.** Given a positive integer x, let U and W be two disjoint sets with $|U| = x$ and $|W| = \infty$. Then

$$\nu(x) = \max_P |V(P)|,$$

where the maximum is taken over all (ℓ, k)-paths P (in the complete k-graph on $U \cup W$) such that

$$U \subset V(P) \subset U \cup W \qquad \text{and} \qquad |e \cap U| \geq k - \ell + 1 \qquad \text{for all} \qquad e \in P. \tag{4}$$

vertex of U). Since $\nu(x)$ is monotone, for any non-negative real number z we can define

$$\mu(z) = \max\{x : \nu(x) \leq z\} \quad \text{and} \quad \mu^*(z) = \mu(z) + 1 = \min\{x : \nu(x) > z\}. \tag{5}$$

In the Appendix we prove several properties of function ν which will be heavily used throughout our proof.

2.2. Parameters setting

In this subsection we define parameters and sets to be used in our construction. Set

$$N_0 := 100k^{10}, \tag{6}$$

let $N \geq N_0$ be an integer divisible by $k - \ell$, and

$$n := \left\lfloor \frac{N}{11k^5} \right\rfloor. \tag{7}$$

It can be easily deduced from (6) and (7) that

$$11k^5 \leq \frac{N}{n} \leq 11.5k^5 \quad \text{and} \quad n \geq N/(11k^5) - 1 \geq 9k^5. \tag{8}$$

Further, recall definitions in (5) and set

$$\begin{aligned} z &:= \frac{N + 4k^3}{n} - (3k - 4\ell), \\ x &:= \mu\,(z) + 2\lfloor k/2 \rfloor, \\ x^* &:= \mu^*\,(z) + 2\lfloor k/2 \rfloor + (k - 2\ell) = x + (k - 2\ell) + 1. \end{aligned} \tag{9}$$

The following tight estimates of N lie at the heart of our construction, which will become evident only at the conclusions of the proofs of the crucial Lemmas 10 and 11. The proof is deferred to the Appendix

Proposition 5. *There exist $x_i \in \{x, x^*\}$, $i = 1, \dots, n$, such that for each $I \subset \{1, \dots, n\}$ with $|I| = n-1$,*

$$(3k - 4\ell)n + \sum_{i \in I} \nu(x_i - 2\lfloor k/2 \rfloor) + 8k^4 < N < (3k - 4\ell)n + \sum_{i=1}^{n} \nu(x_i - 2\lfloor k/2 \rfloor) - 4k^3. \tag{10}$$

Finally, we are ready to define the vertex set of the hypergraphs to be constructed. Let $\{A_i, B_i : i = 1, \dots, 2n\}$ be a family of $4n$ pairwise disjoint sets of sizes

$$|A_i| = \begin{cases} 2\lfloor k/2 \rfloor + \ell, & i = 1, \dots, n \\ 2k - 2\ell - 3, & i = n+1, \dots, 2n, \end{cases} \tag{11}$$

and

$$|B_i| = \begin{cases} x_i - 2\lfloor k/2 \rfloor - \ell, & i = 1, \dots, n \\ b_i & i = n+1, \dots, 2n, \end{cases} \tag{12}$$

where the x_i's are defined via Proposition 5, while the b_i's differ from each other by at most one and are chosen in such a way that

$$\sum_{i=1}^{2n}(|A_i| + |B_i|) = N. \tag{13}$$

The argument that the b_i's are well defined along with some bounds on them, as well as on the x_i's is given in Appendix.

2.3. Main construction

Let G_1 be a maximally non-hamiltonian graph with $V(G_1) = [n] = \{1, \dots, n\}$ and $\Delta(G_1) \le 5$. The existence of such a graph can be deduced for each $n \ge 52$ from the results in [8,9] (see Cor. 2.6 in [4]). Our construction is based on the graph G obtained from G_1 by attaching n vertices $n+1, \dots, 2n$ and n edges $\{i, n+i\}$, $i = 1, \dots, n$, so that each new vertex has degree one.

Fix $2 \le \ell < (k-1)/2$. The desired k-graph H will be defined on an N-vertex set

$$V = \bigcup_{i=1}^{2n} U_i, \quad \text{where} \quad U_i = A_i \cup B_i \tag{14}$$

and A_i, B_i are given in the previous subsection (cf. (13)).

Before defining the edge set of H, we need some more terminology and notation. For a graph F and a set $S \subset V(F)$, denote by $F[S]$ the subgraph of F induced by S. For two k-graphs F_1 and F_2 with $V(F_1) = V(F_2)$, we denote by $F_1 \cup F_2$ the k-graph on the same vertex set whose edge set is the union of the edge sets of F_1 and F_2.

For $S \subset V$, set

$$tr(S) = \{i : S \cap U_i \ne \emptyset\}, \quad tr_1(S) = tr(S) \cap [n], \quad \text{and} \quad \min(S) = \min\{i \in tr(S)\}.$$

Note that $tr_1(S) \subset V(G_1)$. The set $tr(S)$ is sometimes called *the trace of S*.

Further, let $c(S)$ be the number of connected components of $G^3[tr(S)]$, where G^3 is the third power of G, that is, the graph with the same vertex set as G and with edges joining all pairs of distinct vertices which are at distance at most three in G.

We define the desired k-graph H in terms of three other k-graphs, H_1, H_2, and H_3. Let

$$H_1^1 = \left\{ e \in \binom{V}{k} : \exists \{i,j\} \in G_1,\ tr_1(e) = \{i,j\},\ |A_i \cap e| \ge \lfloor k/2 \rfloor \text{ and } |A_j \cap e| \ge \lfloor k/2 \rfloor \right\},$$

$$H_1^2 = \left\{ e \in \binom{V}{k} : \text{for some } i \in [n],\ tr(e) = \{i, n+i\},\ |A_i \cap e| = \ell+1,\ |A_{n+i} \cap e| = k-\ell-1 \right\},$$

and

$$H_1 = H_1^1 \cup H_1^2.$$

Remark 6. Note that when k is odd, for an edge $e \in H_1^1$ one may actually have $tr(e) = \{i, j, r\}$, where $\{i,j\} \in G_1$, $|A_i \cap e| = \lfloor k/2 \rfloor$, $|A_j \cap e| = \lfloor k/2 \rfloor$, and $r \in \{n+1, \dots, 2n\}$, $|U_r \cap e| = 1$. Note also that for an edge $e \in H_1^2$, we have $tr(e) = \{i, n+i\} \in G - G_1$. It follows that $H_1^1 \cap H_1^2 = \emptyset$.

Further, let

$$H_2 = \left\{ e \in \binom{V}{k} : \left| e \cap U_{\min(e)} \right| \ge k - \ell + 1 \right\}.$$

Note that $H_1 \cap H_2 = \emptyset$. Indeed, if $e \in H_1$, then $|e \cap U_{\min(e)}| \le \lceil k/2 \rceil < k - \ell + 1$.

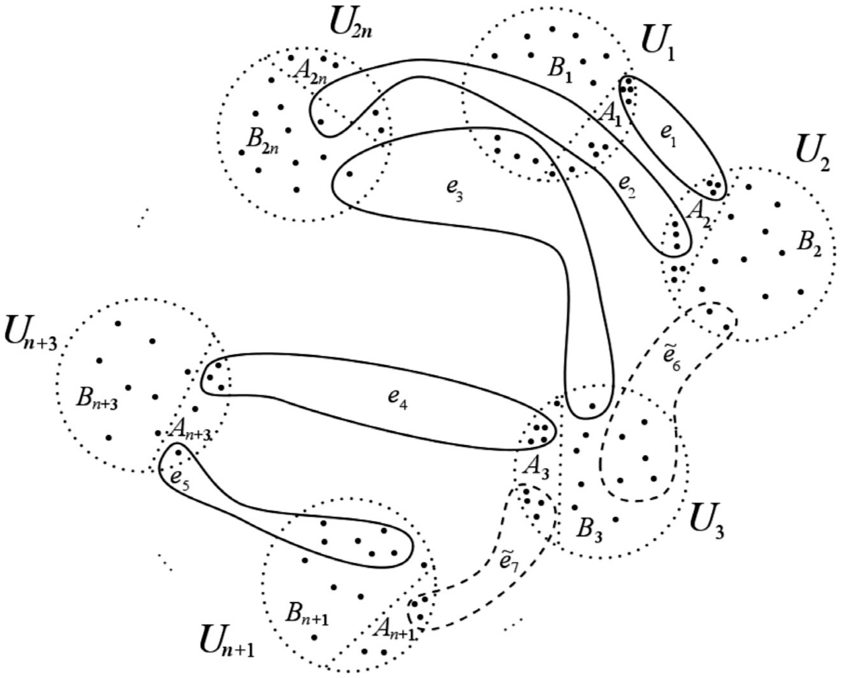

Fig. 1. Illustration of definitions of H_1^1, H_1^2, and H_2 for $k = 7$ and $\ell = 3$: $e_1, e_2 \in H_1^1$, $e_4 \in H_1^2$, $e_3, e_5 \in H_2$, while $\tilde{e}_6, \tilde{e}_7 \notin H_1 \cup H_2$.

Example 7. To illustrate these definitions, let us look at Fig. 1 and the fate of the various edges depicted there. We have $k = 7$ and $\ell = 3$. Assume that $\{1, 2\}$ is an edge of G_1. As $tr_1(e_1) = tr(e_1) = \{1, 2\}$ and $|e_1 \cap A_1| \geq |e_1 \cap A_2| = 3 = \lfloor 7/2 \rfloor$, $e_1 \in H_1^1$. Further, $tr(e_2) = \{1, 2, 2n\}$, but $tr_1(e_2) = \{1, 2\}$. What is more, $|e_2 \cap A_1| = |e_2 \cap A_2| = 3 = \lfloor 7/2 \rfloor$, so $e_2 \in H_1^1$ too.

Since $|e_3 \cap U_1| = 5 = k-\ell+1$ and $\min(e_3) = 1$, we have $e_3 \in H_2$. Similarly, $e_5 \in H_2$. Furthermore, $tr(e_4) = \{3, n + 3\}$, $|e_4 \cap A_3| = 4 = \ell + 1$, and $|e_4 \cap A_{n+3}| = 3 = k - \ell - 1$, so $e_4 \in H_1^2$. Finally, $|\tilde{e}_6 \cap U_3| = 5 \geq k - \ell + 1$, but $\min(e_6) = 2$ and $|\tilde{e}_6 \cap U_2| = 2$. Hence $\tilde{e}_6 \notin H_1 \cup H_2$. Similarly, $\tilde{e}_7 \notin H_1 \cup H_2$.

Recall that

$$p = \max\{\ell, k - 2\ell - 1, \lceil k/2 \rceil - \ell\}. \tag{15}$$

The third element of the construction is

$$H_3 = \left\{ e \in \binom{V}{k} : c(e) \leq p \right\}.$$

Fact 8. *We have $H_1 \cup H_2 \subseteq H_3$.*

Proof. If $e \in H_1$, then $|tr(e)| \leq 3$ and $tr(e)$ contains an edge of G. Thus, $c(e) \leq 2 \leq \ell \leq p$ and $e \in H_3$. If $e \in H_2$, then $|e \cap U_{\min(e)}| \geq k - \ell + 1$ and, consequently, $|tr(e)| \leq 1 + (\ell - 1) = \ell \leq p$. Clearly, $c(e) \leq |tr(e)|$, hence $e \in H_3$ also in this case. □

We are going to show (cf. Lemma 10 in Section 3) that $H_1 \cup H_2$ is non-ℓ-hamiltonian. For each $e \in \binom{V}{k} \setminus H$, let $H + e$ be the hypergraph obtained from H by adding e to its edge set. Taking Lemma 10 for granted and in view of Fact 8, we define H as a non-ℓ-hamiltonian k-graph satisfying the containments

$$H_1 \cup H_2 \subseteq H \subseteq H_3$$

and such that $H + e$ is ℓ-hamiltonian for every $e \in H_3 \setminus H$. (If H_3 is non-ℓ-hamiltonian itself, we set $H = H_3$.)

2.4. Proof of Theorem 2

In [4] (cf. Fact 2.2), we proved the following simple result. Let $comp(F)$ denote the number of connected components of a graph F.

Claim 9 ([4]). *Let r, p, and Δ be constants. If $\Delta(G) \le \Delta$, then the number of r-element subsets $T \subseteq V(G)$ with $comp(G[T]) \le p$ is $O(n^p)$.* □

Theorem 2 is a consequence of Claim 9, the construction of H presented in the previous subsection, and the following two lemmas the proofs of which are deferred to Sections 3 and 4. Lemma 10 guarantees that the definition of H is meaningful.

Lemma 10. *$H_1 \cup H_2$ is non-ℓ-hamiltonian.*

On the other hand, Lemma 11 implies quickly that H is indeed ℓ-hamiltonian saturated (see the proof of Theorem 2 below.)

Lemma 11. *For every $e \in \binom{V}{k} \setminus H_3$, the k-graph $H_1 \cup H_2 + e$ is ℓ-hamiltonian.*

Proof of Theorem 2. As stated in (1), $\mathrm{sat}(N, k, \ell) = \Omega(N^\ell)$. In order to prove the upper bound, we begin by showing that $|H| = O(N^p)$. Observe that

$$H_3 = \bigcup_{T \subset V(G)} \left\{ e \in \binom{V}{k} \; : \; tr(e) = T \right\},$$

where the sum is over all subsets T of $V(G)$ of size at most k with $comp(G^3[T]) \le p$. Since $\Delta(G_1) \le 5$, we have $\Delta(G) \le \Delta_1 + 1 \le 6$ and $\Delta(G^3) \le (\Delta_1 + 1)\Delta_1^2 \le 150$. Thus, by Claim 9 with $r \le k$, the number of such subsets T is $O(n^p)$. Moreover, by (9), (60), (11)–(12) and (63),

$$|U_i| = |A_i| + |B_i| \le \begin{cases} x_i \le x + k \le 12k^5 + k \le 13k^5 & i = 1, \dots, n \\ b_i + 2k \le 12k^5 + 2k \le 13k^5 & i = n+1, \dots, 2n. \end{cases} \tag{16}$$

Hence, given T,

$$\left| \left\{ e \in \binom{V}{k} \; : \; tr(e) = T \right\} \right| \le \binom{\sum_{i \in T} |U_i|}{k} \le (|T| \cdot 13k^5)^k = O(1).$$

Consequently, $|H_3| = O(n^p) = O(N^p)$ and, thus, also $|H| = O(N^p)$.

It remains to show that H is ℓ-hamiltonian saturated. Recall that, by construction (and Lemma 10), H is non-ℓ-hamiltonian. Let $e \in \binom{V}{k} \setminus H$. If $e \in H_3$ then, by the definition of H, $H + e$ is ℓ-hamiltonian. On the other hand, if $e \in \binom{V}{k} \setminus H_3$, then $H + e \supseteq H_1 \cup H_2 + e$ is ℓ-hamiltonian by Lemma 11. This shows that H is, indeed, ℓ-hamiltonian saturated and the proof of Theorem 2 is completed. □

3. Proof of Lemma 10

3.1. (ℓ, k)-paths in $H_1 \cup H_2$

Before turning to the actual proof, we first establish some facts about (ℓ, k)-paths in $H_1 \cup H_2$.

Fact 12. *If P is an (ℓ, k)-path in H_1^2, then P has at most two edges.*

Proof. Suppose there is an (ℓ, k)-path $P = (e_1, e_2, e_3)$ in H_1^2. Then $tr(e_1) \cap tr(e_2) \neq \emptyset$ and $tr(e_2) \cap tr(e_3) \neq \emptyset$. But then, for some j, $tr(e_1) = tr(e_2) = tr(e_3) = \{j, n+j\}$. Since $e_1 \cap e_3 = \emptyset$, it follows that, in particular, $|A_{n+j} \cap e_1| = |A_{n+j} \cap e_3| = k - \ell - 1$ which together exceed the size of A_{n+j} set by the second part of (11). □

Fact 13. *If P is an (ℓ, k)-path in H_2, then there is an index $j \in [2n]$ such that $\min(f) = j$ for every $f \in P$, that is, every edge of P draws at least $k - \ell + 1$ vertices from the same U_j.*

Proof. Let $e, e' \in P$ with $|e \cap e'| = \ell$. Let $j = \min(e)$. Since $|e \cap U_j| \geq k - \ell + 1$, we have $|e' \cap U_j| \geq 1$. Hence, $j \in tr(e')$ and so $\min(e') \leq \min(e)$. By symmetry, $\min(e) \leq \min(e')$. Thus $\min(e') = \min(e) = j$. By transitivity, $\min(f) = j$ for every $f \in P$. □

Claim 14. *Let $s \geq 1$ and let $P = (e, e_1, \dots, e_s, e')$ be an (ℓ, k)-path such that $e, e' \in H_1$ and $e_1, \dots, e_s \in H_2$. Then*

(i) $\min(e_1) = \cdots = \min(e_s) \in tr_1(e) \cap tr_1(e')$;
(ii) $|\{e, e'\} \cap H_1^2| \leq 1$.

Proof. By Fact 13, $\min(e_i) = j$ for some $j \in [2n]$ and every $i = 1, \dots, s$. Since, by definition of H_2, $|e_1 \cap U_j| \geq k - \ell + 1$ and $|e_s \cap U_j| \geq k - \ell + 1$, we have $|e \cap U_j| \geq 1$ and $|e' \cap U_j| \geq 1$ and so, $j \in tr(e) \cap tr(e')$. If, say, $e \in H_1^1$, then $tr(e) \subset [n]$, unless k is odd and $|tr(e)| = 3$. But then, for the unique element $r \in tr(e) \cap \{n+1, \dots, 2n\}$, we have $|e \cap U_r| = 1$ (cf. Remark 6), while, in fact, $|e \cap e_1| \geq 2$. This means that there is $i \in tr_1(e)$ and so, $j \leq i \leq n$ as well.

If, on the other hand, $e, e' \in H_1^2$, then, as $tr(e) \cap tr(e') \neq \emptyset$, for some $i \in [n]$, we have $tr(e) = tr(e') = \{i, n+i\} \ni j$. Thus, by the definition of H_1^2, $|A_{n+j} \cap e| = |A_{n+j} \cap e'| = k - \ell - 1$ which together exceed the size of A_{n+j} set by the second part of (11). This is a contradiction which excludes this case and simultaneously completes the proof of both parts, (i) and (ii). □

Proposition 15. *Let $s \geq 1$ and $P = (e, e_1, \dots, e_s, e')$ be an (ℓ, k)-path in $H_1 \cup H_2$ such that $P \cap H_1^1 = \{e, e'\}$. Then the following hold:*

(a) $P \cap H_1^2 \subset \{e_1, e_s\}$;
(b) If $P \cap H_1^2 = \{e_1, e_s\}$, *then* $s = 2$;
(c) For $i = 1, \dots, s$, *we have* $\min(e_i) \in tr_1(e) \cap tr_1(e')$.

Proof. Since $s \geq 1$ and $\ell < k/2$, we have $e \cap e' = \emptyset$. If $P \cap H_1^2 = \emptyset$, then the statements (a) and (b) are vacuous, while (c) follows from Claim 14(i).

Assume that $P \cap H_1^2 = \{f_1, \dots, f_t\}$, $t \geq 1$, where f_i, $i = 1, \dots, t$, are listed in the order of appearance in P. By Claim 14(ii), $f_1, \dots, f_t$ are consecutive edges of P, while by Fact 12, $t \leq 2$. Recall the definition of H_1^2 and let $tr(f_1) = \{j, n+j\}$ for some $j \in [n]$.

When $t = 2$, noticing that $tr(f_1) \cap tr(f_2) \neq \emptyset$ and remembering the structure of G, we have, in fact, $tr(f_1) = tr(f_2) = \{j, n+j\}$. If $e \cap f_1 \neq \emptyset$, then $j \in tr(e)$. Indeed, otherwise $\left|e \cap U_{n+j}\right| = |e \cap f_1| = \ell \geq 2$, which is not possible by the definition of H_1^1, cf. Remark 6. If $e \cap f_1 = \emptyset$, then, by Claim 14(i) applied to the sub-path of P stretching between e and f_1, we have $j \in tr(e)$ too. Similar argument holds for f_2 and e' implying that $j \in tr(e')$. Thus, $j \in tr(e) \cap tr(e')$. Since $j \leq n$, it means that $j \in tr_1(e) \cap tr_1(e')$.

To prove (a), suppose that $e_i \in H_1^2$ for some $2 \leq i \leq s - 1$. Then, the edges e, e_i, e' are pairwise disjoint. Moreover, by the definitions of H_1^1 and H_1^2, $|A_j \cap e| \geq \lfloor k/2 \rfloor$, $|A_j \cap e'| \geq \lfloor k/2 \rfloor$, and $|A_j \cap e_i| = \ell + 1$, which together exceed the size of A_j set by the first part of (11).

To prove (b), suppose that $e_1, e_s \in H_1^2$ and $s \geq 3$. Then $e_1 \cap e_s = \emptyset$ and, again by the definition of H_1^2, $|A_{n+j} \cap e_1| = |A_{n+j} \cap e_s| = k - \ell - 1$, which together exceed the size of A_{n+j} set by the second part of (11).

It remains to prove part (c). It was already shown above that for every edge $f \in P \cap H_1^2$ we have $j = \min(f) \in tr_1(e) \cap tr_1(e')$. Assume now that $P \cap H_2 \neq \emptyset$. Then, in view of (a) and (b), without loss

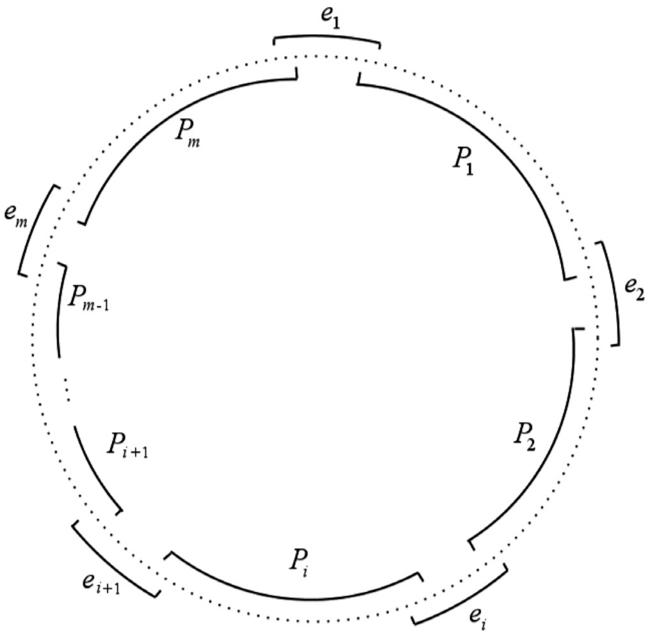

Fig. 2. The structure of phantom C.

of generality we may further assume that $e_1 \in H_1^2$, while $e_2, \dots, e_s \in H_2$. By Claim 14(i) applied to the path from e_1 to e', we conclude that for each $f \in P \cap H_2$, we have $\min(f) \in tr_1(e_1) = \{j\}$, as well as, $\min(f) \in tr_1(e')$. Hence, $\min(f) = j \in tr_1(e) \cap tr_1(e')$ and (c) holds, indeed, for all inner edges of P. □

3.2. Proof of Lemma 10 — the structure of phantom C.

Suppose C is a hamiltonian (ℓ, k)-cycle in $H_1 \cup H_2$. We are going to show that $|V(C)| < N$ which will be a contradiction. Our proof at some point (cf. proof of Claim 17) relies on the assumption that the graph G_1 is not hamiltonian.

We first consider the case when $C \cap H_1^1 = \emptyset$. Then, by Fact 12 and Claim 14(ii), C consists of at most two intersecting edges from H_1^2 and a path $P \subset H_2$. By Fact 13, the bound (16) on $|U_j|$, and Definition 4 of function ν with $U = U_j$, we have, using also Proposition 23(b) and formula (6),

$$|V(C)| \le 2k - 3\ell + \nu(13k^5) \le 2k + 13k^6 < N_0 \le N.$$

From now on we may thus assume that $C \cap H_1^1 \neq \emptyset$. Let $M = \{e_1, \dots, e_m\}$, $m \ge 1$, be a maximal set of pairwise disjoint edges of $C \cap H_1^1$, listed in the order of appearance on C. Further, for $i = 1, \dots, m$, let P_i be the (ℓ, k)-path in C joining the last ℓ vertices of e_i with the first ℓ vertices of e_{i+1}, where $e_{m+1} := e_1$. Notice that

$$C \setminus M = \bigcup_{i=1}^{m} P_i, \tag{17}$$

where all P_i's are vertex disjoint (see Fig. 2).

Let l_i be the first edge of P_i and r_i be the last edge of P_i (note that they may coincide). We also define P_i' to be the (ℓ, k)-path arising from P_i by removing l_i and r_i. Observe that, by the definition of M,

$$P_i' \subset H_1^2 \cup H_2, \tag{18}$$

Also, $P_i' = \emptyset$ if $l_i = r_i$, and then the number of vertices between e_i and e_{i+1} is $k - 2\ell$. Since for a non-empty P_i' the number of vertices between e_i and the beginning of P_i', as well as, between the end of P_{i-1}' and e_i, is exactly $k - 2\ell$, by (17) we have

$$|V(C)| \le m(3k - 4\ell) + \sum_{i=1}^{m} |V(P_i')|. \tag{19}$$

In view of this, in order to show that $|V(C)| < N$, our plan is to utilize the left inequality in (10). This, in turn, will require us to set strong bounds on m and $|V(P_i')|$.

Beginning with the former task, recall that for each $e \in H_1^1$, $tr_1(e)$ consists of exactly one edge of G_1. These edges may, however, repeat for various e's, so that

$$Tr(M) := \{tr_1(e) : e \in M\}$$

is a multigraph of size m on vertex set $[n]$. Since, for each $e \in M$ and $j \in tr_1(e)$, $|e \cap A_j| \ge \lfloor k/2 \rfloor$, it follows by the first part of (11) that

$$\Delta(Tr(M)) \le 2, \tag{20}$$

and, in particular,

$$m \le n. \tag{21}$$

To improve this bound, we distinguish between nice and problematic paths P_i. Observe that each edge $e \in \left(H_1^1 \cap C\right) \setminus M$ intersects some $e_i \in M$, so $e = l_i$ or $e = r_{i-1}$. We call an edge l_i or r_i *bad* if it belongs to H_1^1, $|P_i| \ge 2$, and, resp., $tr_1(l_i) \ne tr_1(e_i)$ or $tr_1(r_i) \ne tr_1(e_{i+1})$. We call P_i *problematic* if either l_i or r_i is bad, or $P_i' \cap H_1^2 \ne \emptyset$. Otherwise, we call P_i *nice*. In particular, if P_i is problematic, then $|P_i| \ge 2$ and $l_i \ne r_i$. Let q be the number of problematic (ℓ, k)-paths among $P_1, \dots, P_m$.

We next show that the presence of problematic paths makes the number of edges in $Tr(M)$ smaller.

Claim 16.

$$m \le n - \frac{1}{2}\left\lceil \frac{q}{k} \right\rceil \tag{22}$$

Proof. Recall (20). We are going to show that problematic paths cause some vertices to have degrees smaller than 2 which will lead to the improvement (22) over (21). Let $P := P_i$ be problematic and assume first that there is a bad edge, say l_i, in P. Then $tr_1(l_i) \ne tr_1(e_i)$ and, consequently, by considering separately the cases when $tr_1(l_i) \cap tr_1(e_i) = \emptyset$ and when $|tr_1(l_i) \cap tr_1(e_i)| = 1$, there exists vertex $j := j_i \in tr_1(l_i)$ such that $j \notin tr(e_i)$ (recall Remark 6 that one might have $|tr(e_i)| = 3$). Thus, by the definition of H_1^1, we have $\left|\left(l_i \cap A_j\right) \setminus e_i\right| \ge \lfloor k/2 \rfloor$. Since also $|P| \ge 2$, we have $l_i \cap e_{i+1} = \emptyset$. And, obviously, by construction, l_i is disjoint from all other edges in M. Thus, in fact,

$$\left|\left(l_i \cap A_j\right) \setminus (e_1 \cup \dots \cup e_m)\right| \ge \lfloor k/2 \rfloor . \tag{23}$$

By symmetry, (23) holds if r_i is a bad edge of P.

Another reason for P_i being problematic might be that P_i' contains an edge $f := f_i \in H_1^2$. Then, by the definition of H_1^2, there exists a vertex $j := j_i \in tr_1(f)$ such that $\left|f \cap A_j\right| = \ell + 1$. Since in this case f does not intersect any edge of M, $f \cup l_i \cup r_i \subset V(P_i)$, we may conclude that, for each $i = 1, \dots, m$ for which P_i is problematic, there exists $j_i \in tr_1(V(P_i))$ such that

$$\left|(V(P_i) \cap A_{j_i}) \setminus (e_1 \cup \dots \cup e_m)\right| \ge \ell + 1. \tag{24}$$

As $|A_{j_i}| = 2\lfloor k/2\rfloor + \ell$, inequality (24) and the definition of H_1^1 imply that $deg_{Tr(M)}(j_i) \le 1$. The j_i's need not be different. However, at most

$$\frac{|A_j|}{\ell+1} \le \frac{2\lfloor k/2\rfloor + \ell}{\ell+1} \le 1 + 2\lfloor k/2\rfloor - 1 \le k$$

problematic paths P_i's may yield the same j for which A_j satisfies (24). Thus, at least $\lceil q/k\rceil$ different vertices $j \in [n]$ have $deg_{Tr(M)}(j_i) \le 1$. Therefore,

$$\sum_{i=1}^{n} deg_{Tr(M)}(i) \le 2n - \left\lceil \frac{q}{k} \right\rceil$$

and, consequently,

$$m = |Tr(M)| \le n - \frac{1}{2}\left\lceil \frac{q}{k} \right\rceil. \quad \square$$

In view of Claim 16, we have $m \le n-1$ for $q \ge 1$. Now we will get a similar improvement over $m \le n$ in the case when no problematic paths are present (unless, for some i, $P_i' = \emptyset$, which is, anyhow, to our advantage).

Claim 17. *Suppose that $P_i' \neq \emptyset$ for every $i = 1, \dots, m$. Then*

$$m \le n - 1. \tag{25}$$

Proof. If $q \ge 1$, then (25) follows by Claim 16. Assume that $q = 0$ and suppose that $|Tr(M)| = m = n$. Then, by (20), $Tr(M)$ is a 2-regular spanning subgraph of G_1, with possibly some parallel edge of multiplicity 2. We aim at showing that $Tr(M)$ is connected. Since $q = 0$, each P_i is nice and so, by (18), $P_i' \subset H_2$.

Let j be an index guaranteed by Fact 13 applied to P_i'. Further, let $\bar{P}_i$ be the shortest extension of the path P_i' within C whose both end-edges belong to H_1^1. Then, by Proposition 15(c) applied to $\bar{P}_i$, the traces of its end-edges contain $j \in [n]$. So, if e_i is one of these end-edges, we then have $j \in tr_1(e_i)$. Otherwise, that is, when $l_i \in H_1^1$ and, thus, l_i is an end-edge of $\bar{P}_i$, we have $j \in tr_1(l_i)$. However, since P_i is nice, l_i is not bad and so, $tr_1(e_i) = tr_1(l_i)$. Hence, $j \in tr_1(e_i)$, anyway. By symmetry, $j \in tr_1(e_{i+1})$, too. This means, however, that $Tr(M)$ is connected and, consequently, $Tr(M)$ is a hamiltonian cycle in G_1, a contradiction with the choice of G_1. $\square$

3.3. *Proof of Lemma 10 — the length of phantom C.*

So far we have expressed the presumed hamiltonian (ℓ, k)-cycle C in the form (17) and set bounds on $m = |M|$ (see Claims 16 and 17). In order to take advantage of (19), we also need to estimate $\left|V(P_i')\right|$. We do it separately for nice and problematic paths. Recall Definition 4 of function ν from Section 2.1.

Claim 18. *If P_i is nice, then for some $j := j_i \in [n]$,*

$$\left|V(P_i')\right| \le \nu\left(x_j - 2\lfloor k/2\rfloor\right).$$

Proof. Since P_i is nice, $P_i' \subset H_2$ by (18). If $P_i' = \emptyset$, then the claim trivially holds. Let $f \in P_i'$ and $j = \min(f)$. Similarly, as in the proof of Claim 17, we infer that $j \in tr_1(e_i)$ and $j \in tr_1(e_{i+1})$. Thus, $|A_j \cap e_i| \ge \lfloor k/2\rfloor$ and $|A_j \cap e_{i+1}| \ge \lfloor k/2\rfloor$, which implies that $\left|V(P_i') \cap U_j\right| \le x_j - 2\lfloor k/2\rfloor$. Therefore, the claim follows by Fact 13 and Definition 4 of ν with $U = V(P_i') \cap U_j$. $\square$

Claim 19. *If P_i is problematic, then for some $j := j_i \in [n]$,*

$$\left|V(P_i')\right| \le \nu(x_j) + k/2.$$

Proof. Let P_i'' be the shortest subpath of P_i with both end-edges belonging to H_1^1. By the choice of M, P_i'' exists and satisfies $P_i' \subset P_i'' \subset P_i$. By Proposition 15(a,b) applied to P_i'', $|P_i''| \le 4$ or P_i'' contains at most one edge of H_1^2. In the former case the claimed inequality holds, because $|V(P_i')| < 4k$, while, by (59), $\nu(x_j) \ge x_j \ge 10k^4$. In the latter, P_i' contains at most one edge of H_1^2, as well. Moreover, this edge, if exists, is either the first or the last edge of P_i'. Say, it is the first. Then the rest of P_i' (i.e., P_i' minus the first or the last $\ell \le k/2$ vertices) is contained in H_2 and either $r_i \in H_1^1$, or $r_i \in H_2$ (recall that since P_i is problematic, $r_i \ne l_i$). Hence, by Claim 14(i), applied to an appropriate extension of P_i', there exists $j \in [n]$ such that $j = \min(f)$ for all $f \in P_i' \cap H_2$. Thus, $\left|V(P_i') \cap U_j\right| \le |U_j| = x_j$ and the claim follows again by Fact 13 and Definition 4. □

We are now in the position to finish the proof of Lemma 10. Suppose that there are exactly q problematic paths among the P_i's. Let $I \subset [1, m]$ be the set of those indices i for which $P_i' \ne \emptyset$. Further, let $I' \subset I$ be the set of those indices i for which P_i is problematic, and $I'' = I \setminus I'$. By (19), Claims 18 and 19, and (49),

$$\begin{aligned}|V(C)| &\le m(3k-4\ell) + \sum_{i\in I}|V(P_i')|\\ &\le m(3k-4\ell) + \sum_{i\in I'}(\nu(x_{j_i}) + k/2) + \sum_{i\in I''}\nu(x_{j_i} - 2\lfloor k/2\rfloor)\\ &\le m(3k-4\ell) + \sum_{i\in I'}(\nu(x_{j_i} - 2\lfloor k/2\rfloor) + k^2 + k/2) + \sum_{i\in I''}\nu(x_{j_i} - 2\lfloor k/2\rfloor)\\ &= m(3k-4\ell) + \sum_{i\in I}\nu(x_{j_i} - 2\lfloor k/2\rfloor) + (k^2+k/2)q.\end{aligned}$$

If $q = 0$, then, by Claim 17, either $m \le n-1$ or $|I| \le n-1$, so we have $|V(C)| \le m(3k-4\ell) + \sum_{i\in I}\nu(x_{j_i} - 2\lfloor k/2\rfloor)$. If $q \ge 1$, then, by Claim 16, $m \le n - \frac{1}{2}\left\lceil \frac{q}{k}\right\rceil$. So, every increase of q by $2k$ forces a decrease of m by 1. However, since by (59), $\nu(x_{j_i} - 2\lfloor k/2\rfloor) \ge x_{j_i} - 2\lfloor k/2\rfloor > 10k^4 - k > 9k^4$, the maximum is attained when m is as large as possible, that is, for $m = n-1$ and $q = 2k$. Hence, in either case,

$$|V(C)| \le n(3k-4\ell) + \sum_{i\in I}\nu(x_{j_i} - 2\lfloor k/2\rfloor) + 2k(k^2+k/2), \tag{26}$$

where $I \subset [1, n]$ with $|I| \le n-1$. Combined with the left inequality in (10), this yields, with some margin, that $|V(C)| < N$, and so C cannot be a hamiltonian (ℓ, k)-cycle, a contradiction. □

4. Proof of Lemma 11

4.1. *The idea of the proof*

In the proof of Lemma 10 we supposed that there was a hamiltonian (ℓ, k)-cycle C in $H_1 \cup H_2$ and got a contradiction by showing that it would be too short to cover all N vertices. Now, we have at disposal just one more edge e which, however, will make all the difference. In fact, despite the opposite goals these two proofs bear some similarities.

In the former proof we represented C as a concatenation of several paths in H_2 joint together via short paths centered at edges of H_1^1. A crucial ingredient of that proof was to show that there are no more than $n-1$ disjoint edges in $H_1^1 \cap C$, causing the whole cycle to be too short.

Now, we will turn that idea around and *construct* a hamiltonian (ℓ, k)-cycle in $H_1 \cup H_2 + e$, by constructing n disjoint (ℓ, k)-paths $P_1, \dots P_n$ in H_2 and joining them by disjoint sequences of vertices $Q_0, \dots, Q_{n-1}$ (let us call them *bridges* from now on), built around edges of H_1. In fact, for technical reasons, in the forthcoming proof we will first build the bridges $Q_0, \dots, Q_{n-1}$ and only then the paths $P_1, \dots, P_n$. The reason there were less than n bridges in the proof of Lemma 10 was that G_1 was not hamiltonian. On the other hand, G_1 is *maximally* non-hamiltonian and the new edge $e \notin H$ will bring about the missing bridge (Q_0). This will be done by a clever choice of two vertices of $tr(e)$.

4.2. The choice of i and j

Let us fix $e \in \binom{V}{k} \setminus H_3$. Recall that, by the definition of H_3, $c(e) \geq p+1$, where p was defined in (15). We are going to choose carefully two vertices, i and j, in $tr(e)$. They have to come from different components of $G^3[tr(e)]$. In particular, $ij \notin G$. Even more, if $i = n + i'$ or $j = n + j'$ for some $1 \leq i', j' \leq n$, then also, respectively, $ij', i'j, i'j' \notin G_1$. (This is, in fact, why we considered components in $G^3[tr(e)]$, and not just in $G[tr(e)]$.) The bottom line is that, due to being maximally non-hamiltonian, G_1 possesses a hamiltonian path connecting i (or its unique neighbor) with j (or its unique neighbor). We will ultimately build a hamiltonian (ℓ, k)-cycle in $H_1 \cup H_2 + e$ by following this path in G_1.

Let $C_1, \ldots, C_r$ be connected components of $G^3[tr(e)]$. Further, let

$$\rho(C_t) = \max\{|e \cap U_v| : v \in V(C_t)\}, \quad t = 1, \ldots, r.$$

Without loss of generality we may assume that

$$\rho(C_1) \geq \rho(C_2) \geq \cdots \geq \rho(C_r).$$

We now choose i and j. If $\rho(C_1) \leq \ell$, then $i = \min(e)$. Otherwise, let $i \in V(C_1)$ be such that

$$|e \cap U_i| = \rho(C_1) \geq \ell + 1.$$

Let X be the vertex set of this component of $G^3[tr(e)]$ which contains vertex i (e.g., $X = V(C_1)$ in the latter case) and let $Y = tr(e) \setminus X$. Set

$$e_X = e \cap \bigcup_{v \in X} U_v \quad \text{and} \quad e_Y = e \cap \bigcup_{v \in Y} U_v.$$

Clearly,

$$e = e_X \cup e_Y. \tag{27}$$

Further, if $\rho(C_2) \leq \ell$, then $j = \min(e_Y)$. Otherwise, let $j \in V(C_2)$ be such that

$$|e \cap U_j| = \rho(C_2) \geq \ell + 1.$$

Note that in the latter case $X = V(C_1)$, so, indeed, i and j always belong to different components of $G^3[tr(e)]$.

Now we establish upper bounds on the cardinalities of some parts of e. Since $c(e) \geq p+1$,

$$|e \cap U_t| \leq k - p \text{ for every } t \in tr(e), \tag{28}$$

and, in particular,

$$|e_X| \leq k - p. \tag{29}$$

Note that, by (27) and (29), we also have $e(Y) \geq p$. Inequality (28) can be improved in most cases.

Fact 20. *If $t \in tr(e) \setminus \{i, j\}$, then*

$$|e \cap U_t| \leq \ell.$$

Proof. If $\rho(C_1) \leq \ell$ then the claim is obvious. Suppose $\rho(C_1) \geq \ell + 1$. Thus, $|e \cap U_i| \geq \ell + 1$. If $t \in X \setminus \{i\}$, then, by (15) and (29),

$$|e \cap U_t| \leq k - p - |e \cap U_i| \leq k - p - (\ell + 1) \leq \ell.$$

Let $t \in tr(e) \setminus X = Y$. If $\rho(C_2) \leq \ell$, then, again, the claim is obvious. So, suppose $\rho(C_2) \geq \ell + 1$. Hence, $|e \cap U_j| \geq \ell + 1$. Note that since $|tr(e)| \geq c(e) \geq p + 1$, we have $|tr(e) \setminus \{i, j, t\}| \geq p - 2$, and so

$$|e \cap (U_i \cup U_j \cup U_t)| \leq k - p + 2.$$

Thus, again by (15),

$$|e \cap U_t| \le k-p+2-|e \cap U_i|-|e \cap U_j| \le k-p+2-2(\ell+1) \le 1 < \ell. \quad \square$$

4.3. Construction of bridge Q_0

The construction of Q_0 is based on the extra edge e and the choice of i and j from $tr(e)$. Let us order the vertices of e so that, going from left to right, it begins with all vertices of $e \cap U_j$, followed by all remaining vertices of $e(Y)$. Symmetrically, going from right to left, it begins with all vertices of $e \cap U_i$, followed by the remaining vertices of $e(X)$.

We first we construct an (ℓ, k)-path Q_0' which is the main part of Q_0. We consider four cases with respect to i and j, which, owing to symmetry, reduce to just two (with two further subcases in one of them).

Notation for diagrams. The forthcoming constructions will be illustrated by diagrams in which the following notation is applied. Recall that for each $s = 1, \dots, 2n$, $U_s = A_s \cup B_s$. Any vertex of A_s will be represented by the symbol a_s. Similarly, b_s will stand for any vertex of B_s, while u_s for any vertex of U_s. The asterisk $*$ will fill in for any vertex of $V = \bigcup_{s=1}^{2n} U_s$, or, on one occasion, of $\bigcup_{s=n+1}^{2n} B_s$. Moreover, all vertices appearing in the diagrams will be distinct.

Suppose first that $i, j \in \{1, \dots, n\}$. Let Q_0' be a 3-edge (ℓ, k)-path with the edge e in the middle and two edges e' and e'' from H_2. The first edge e' of Q_0' begins with $k-\ell$ vertices of B_j and ends with the first ℓ vertices of e, while the last (third) edge e'' of Q_0' begins with the last ℓ vertices of e and ends with $k-\ell$ vertices of B_i (see diagram (30) below).

$$Q_0' = \underbrace{b_j \dots b_j}_{k-\ell}\underbrace{\overbrace{u_j * *}^{e_Y}\,\overbrace{*u_i}^{e_X}}_{e}\underbrace{b_i \dots b_i}_{k-\ell}. \tag{30}$$

Recall that either $j = \min(e_Y)$ or $|U_j \cap e| \ge \ell + 1$. Consequently, in each case $\min(e') = j$ and $|e' \cap U_j| \ge k-\ell+1$, so $e' \in H_2$. Similarly, $e'' \in H_2$.

If $i = n+i'$, then we modify the right end of Q_0' as follows. If $|e \cap A_i| \le k-\ell-2$, then we replace the last ℓ vertices of e'' with $k-\ell-1$ vertices of A_i, followed by $\ell+1$ vertices of $A_{i'}$ (see the R-H-S of diagram (31)).

$$Q_0' = \underbrace{b_j \dots b_j}_{k-\ell}\underbrace{\overbrace{u_j * *}^{e_Y}\,\overbrace{*u_i}^{e_X}}_{e}\underbrace{b_i \dots b_i}_{k-2\ell}\underbrace{a_i \dots a_i}_{k-\ell-1}\underbrace{a_{i'} \dots a_{i'}}_{\ell+1}. \tag{31}$$

This way, edge e'' is replaced by edges $e_1'' \in H_2$ and $e_2'' \in H_1^2$. Since $|e \cap A_i| \le k-\ell-2$, we have, indeed, at least $(2k-2\ell-3)-(k-\ell-2) = k-\ell-1$ vertices of A_i available. (As for $A_{i'}$, by (11), $|A_{i'}| \ge k-1+\ell$, and only at most $k-2$ vertices of $A_{i'}$ may belong to e.)

If $|e \cap A_i| \ge k-\ell-1$, we modify Q_0' as indicated in the R-H-S of diagram (32).

$$Q_0' = \underbrace{b_j \dots b_j}_{k-\ell}\underbrace{\overbrace{u_j * *}^{e_Y}\,\overbrace{*a_i \dots a_i}^{e_X}}_{e}\underbrace{a_i \dots a_i}_{k-2\ell-1}\underbrace{a_{i'} \dots a_{i'}}_{\ell+1}. \tag{32}$$

Note that now, again, we have just one edge to the right of e and this is an edge of H_1^2. Furthermore, by (15) and (28),

$$|Q_0' \cap A_i| \le k-p+k-2\ell-1 \le 2k-2\ell-3,$$

so, this construction is feasible.

The case $j = n+j'$ is analogous. In summary, depending on the case, the path Q_0' consists of three to five edges, all contained in $H_1 \cup H_2 + e$. To simplify further notation, from now on, let us assume (w.l.o.g.) that $i \in \{1, n+1\}$ and $j \in \{n, 2n\}$. In fact, we may arbitrarily renumber vertices $1, \dots, n$ and, accordingly, vertices $n+1, \dots, 2n$. Since in the rest of the construction we are going to use only edges e' of H_2 that intersect exactly one of the sets U_i with $1 \le i \le n$, such a renumbering will not affect the sets $U_{\min(e')}$ (which are crucial for the edges of H_2), regardless of how may sets U_i with $n+1 \le i \le 2n$ are intersected by e'.

We complete the construction of Q_0 by adding $k-2\ell$ new vertices from B_n on the left of Q_0' and $k-2\ell$ new vertices from B_1 on the right of Q_0', that is,

$$Q_0 = \underbrace{b_n \dots b_n}_{k-2\ell} Q_0' \underbrace{b_1 \dots b_1}_{k-2\ell}. \tag{33}$$

Note that $k-2\ell \ge 1$ and that Q_0 always begins with at least $k-\ell+1$ vertices from U_n and ends with at least $k-\ell+1$ vertices from U_1. Also, technically, Q_0 is not an (ℓ, k)-path as at either end it is, on purpose, "unfinished".

Before continuing with the construction, let us summarize how many vertices have been taken by Q_0 from each set A_t, $t \in [n]$. To this end, let us partition the set $[n]$ into two subsets

$$T_1 = \{t \in [n] : t \notin tr(e) \text{ and } n+t \notin tr(e)\},$$
$$T_2 = [n] \setminus T_1 \tag{34}$$

and observe that

$$T_1 \subseteq [2, n-1] \quad \text{and} \quad |T_2| \le |tr(e)| \le k. \tag{35}$$

Trivially, by the construction of Q_0, for all $t \in T_1$,

$$(U_t \cup U_{n+t}) \cap Q_0 = \emptyset. \tag{36}$$

Fact 21.

$$|Q_0 \cap A_t| \le \begin{cases} k-p & \textit{for} \quad t \in \{1, n\}, \\ \ell & \textit{for} \quad t \in T_2 \cap [2, n-1], \\ 0 & \textit{for} \quad t \in T_1. \end{cases} \tag{37}$$

Proof. If $t \in T_1$ then the statement follows from (36). If $t \in T_2 \cap [2, n-1]$, then by the construction of Q_0,

$$Q_0 \cap A_t \subseteq Q_0 \cap U_t = e \cap U_t$$

and the second line of (37) holds by Fact 20.

Let $t = 1$. If $i = 1$, then the R-H-S of Q_0' is like in diagram (30), and so, by (28),

$$|Q_0 \cap A_1| = |e \cap A_1| \le |e \cap U_1| \le k-p.$$

If, on the other hand, $i = n+1$, then consider two cases with respect to whether $1 \in tr(e)$ or not.

If $1 \notin tr(e)$, then by diagrams (31) or (32), and by (15),

$$|Q_0 \cap A_1| = \ell + 1 \le k-p.$$

(To see the last inequality one has to check all 3 cases for p.)

On the other hand, if $1 \in tr(e)$, the procedure selecting i implies that

$$|e \cap U_{n+1}| = \rho(C_1) \ge \ell + 1.$$

Furthermore, as 1 and $n+1$ are two vertices of the same component of G, and thus of G^3, we have $\{1, n+1\} \subseteq X$ and, by (29),

$$|e \cap U_1| + |e \cap U_{n+1}| \le |e_X| \le k-p. \tag{38}$$

Hence, again by diagrams (31) or (32),

$$|Q_0 \cap A_1| \le |e \cap U_1| + (\ell+1) \le |e \cap U_1| + |e \cap U_{n+1}| \le k-p.$$

The proof for $t = n$ is analogous, except that in the case $j = 2n$, $n \in tr(e)$, to get an analog of (38), instead of (29) we use the inequality $|tr(e) \setminus \{n, 2n\}| \ge c(e) - 1 \ge p$ which immediately implies that

$$|e \cap U_n| + |e \cap U_{2n}| \le k - p. \quad \square$$

4.4. Construction of bridges $Q_1, \dots, Q_{n-1}$

Since G_1 is maximally non-hamiltonian and $1n \notin G_1$, there is a hamiltonian path in G_1 which begins at vertex 1 and ends at vertex n. W.l.o.g., we assume that its vertex sequence is $1, 2, 3, \dots, n-1, n$. Based on this hamiltonian path we will build a hamiltonian (ℓ, k)-cycle in H.

First, we construct $n-1$ pairwise disjoint edges, $e_1 \dots, e_{n-1} \in H_1$, such that they are also disjoint from e and for each $t = 1, \dots, n-1$, e_t contains $\lfloor k/2 \rfloor$ vertices from A_t followed, if k is odd, by one vertex from $\bigcup_{s=n+1}^{2n} B_s$ and then $\lfloor k/2 \rfloor$ vertices from A_{t+1} (see the diagram below).

$$e_t = \underbrace{a_t \dots a_t}_{\lfloor k/2 \rfloor}(*)\underbrace{a_{t+1} \dots a_{t+1}}_{\lfloor k/2 \rfloor}.$$

Thus, for each $s = 2, \dots, n-1$ we need $2\lfloor k/2 \rfloor$ vertices of A_s which is feasible by (11) and (37), while for $s \in \{1, n\}$ we only need $\lfloor k/2 \rfloor$ vertices of A_s, which is again possible by (11) and (37), and the definition of p in (15).

Next we set aside pairwise disjoint $(k-2\ell)$-element sequences of vertices $L_1, \dots, L_{n-1}$ and $R_1, \dots, R_{n-1}$ which are also disjoint from $Q_0 \cup e_1 \cup \dots \cup e_{n-1}$ and such that for all $t = 1, \dots n-1$ we have $L_t \subset B_t$, while

$$R_t \subset A_{n+t+1} \quad \text{if} \quad t+1 \in T_1\,,$$
$$R_t \subset B_{t+1} \quad \text{if} \quad t+1 \in T_2\,,$$

which is feasible by (11) together with (36), and (12) together with (59), and the bound $|Q_0 \cap B_t| \le |Q_0| < 7k$. Finally, for all $t = 1, \dots n-1$ set

$$Q_t = L_t, e_t, R_t,$$

that is,

$$Q_t = \begin{cases} \underbrace{b_t \dots b_t}_{k-2\ell}\underbrace{a_t \dots a_t}_{\lfloor k/2 \rfloor}(*)\underbrace{a_{t+1} \dots a_{t+1}}_{\lfloor k/2 \rfloor}\underbrace{a_{n+t+1} \dots a_{n+t+1}}_{k-2\ell} & \text{if} \quad t+1 \in T_1\,, \\ \underbrace{b_t \dots b_t}_{k-2\ell}\underbrace{a_t \dots a_t}_{\lfloor k/2 \rfloor}(*)\underbrace{a_{t+1} \dots a_{t+1}}_{\lfloor k/2 \rfloor}\underbrace{b_{t+1} \dots b_{t+1}}_{k-2\ell} & \text{if} \quad t+1 \in T_2\,. \end{cases} \tag{39}$$

So far we have constructed all bridges. Let us summarize how many vertices of each set U_t, $t \in [n]$, were consumed by them. In addition, for future purposes, we are also interested in the usage of A_{n+t}, $t \in T_1$. Let $Q = \bigcup_{t=0}^{n-1} Q_t$ (here Q_t's are understood as sets, not sequences).

Fact 22. *We have the following bounds.*

(i) For each $t \in T_1$, $|Q \cap A_t| = 2\lfloor k/2 \rfloor$, $|Q \cap B_t| = k - 2\ell$, and $|Q \cap A_{n+t}| = k - 2\ell$.
(ii) For each $t \in T_2$, $|Q \cap U_t| \le 2\lfloor k/2 \rfloor + 4k$.

Proof. In general, $Q \cap U_t = (Q_0 \cap U_t) \cup (Q_t \cap U_t) \cup (Q_{t-1} \cap U_t)$, where we assume $Q_n = \emptyset$ for convenience. By (36), when $t \in T_1$, we have $Q_0 \cap U_t = \emptyset$ and $Q_0 \cap A_{n+t} = \emptyset$. Also then, by inspecting (39), $|Q_t \cap A_t| = \lfloor k/2 \rfloor$ and $|Q_t \cap B_t| = k - 2\ell$, while $|Q_{t-1} \cap A_t| = \lfloor k/2 \rfloor$, $|Q_{t-1} \cap B_t| = 0$ and $|Q_{t-1} \cap A_{n+t}| = k - 2\ell$. This proves part (i).

When $t \in [2, n-1] \cap T_2$, we have $|Q_0 \cap U_t| = |e \cap U_t| \le \ell$ by Fact 20, and, again by inspection, $|Q_t \cap U_t| = |Q_{t-1} \cap U_t| = \lfloor k/2 \rfloor + k - 2\ell$, so, altogether, $|Q \cap U_t| \le 2\lfloor k/2 \rfloor + 2(k - 2\ell) + \ell \le 2\lfloor k/2 \rfloor + 4k$.

Consider now the case $t = 1$. Then $i = 1$ or $i = n+1$. If $i = 1$, then bounding trivially $|e \cap U_1| \le k$, by (30) and (33), we have $|Q_0 \cap U_1| \le k + (k - \ell) + (k - 2\ell)$. This, together with

$|Q_1 \cap U_1| = \lfloor k/2 \rfloor + k - 2\ell$, yields that

$$|Q \cap U_1| \le \lfloor k/2 \rfloor + 4k - 5\ell \le \lfloor k/2 \rfloor + 4k.$$

If $i = n + 1$, then by (31), (32) (with $i' = 1$), (33) and (39), and again bounding $|e \cap U_1| \le k$, we obtain

$$|Q \cap U_1| \le (\ell + 1) + (k - 2\ell) + (\lfloor k/2 \rfloor + k - 2\ell) + 4 = \lfloor k/2 \rfloor + 3k - 3\ell + 1 \le \lfloor k/2 \rfloor + 4k.$$

The case $t = n$ is very similar. □

4.5. Construction of paths $P_1, \dots, P_n$

Next, we construct n pairwise vertex disjoint (ℓ, k)-paths $P_t \subseteq H_2$, $t = 1, \dots, n$, such that each P_t consists of all vertices from $U_t \setminus Q$ and some vertices from $\bigcup_{s=n+1}^{2n} U_s \setminus Q$, so that together with the sequences $Q_0, \dots, Q_{n-1}$ they exhaust all N vertices and, after some mending, will yield the ultimate hamiltonian (ℓ, k)-cycle.

By the definition of H_2 and Fact 13, each edge $f \in P_t$ will have to satisfy $\min(f) = t$ and $|f \cap (U_t \setminus Q)| \ge k - \ell + 1$. We are going to build the paths $P_1, \dots, P_t$, in two stages.

Abstract construction

First, instead of $\bigcup_{s=n+1}^{2n} U_s$, we use vertices from some (abstract and disjoint from V) infinite set W and construct paths $P'_1, \dots, P'_t$ which are as large as possible and each edge $f \in P'_t$ satisfies $|f \cap (U_t \setminus Q)| \ge k - \ell + 1$. By Definition 4 of function ν with $U = U_t \setminus Q$ we have $|V(P'_t)| = \nu(|U_t \setminus Q|)$. It will turn out that the total length of these paths and the sequences $Q_0, \dots, Q_{n-1}$ exceeds N, so in the second stage we will truncate them to the total length N (by removing some vertices of W) and, finally, replace the remaining vertices of W by those in $\bigcup_{s=n+1}^{2n} U_s$, obtaining the desired paths $P_1, \dots, P_t$.

We first estimate the lengths of the paths $P'_1, \dots, P'_t$. By Fact 22(i), (11), and (12), for $t \in T_1$ we have $|U_t \setminus Q| = x_t - (2\lfloor k/2 \rfloor + k - 2\ell)$. Thus, by (57) and (58),

$$|V(P'_t)| = \nu\left((x_t - 2\lfloor k/2 \rfloor) - (k - 2\ell)\right) = \nu(x_t - 2\lfloor k/2 \rfloor) \text{ if } t \in T_1 \tag{40}$$

Similarly (but understandably with less precision), by Fact 22(ii), (11), (12), and (49), we have

$$|V(P'_t)| \ge \nu\left((x_t - 2\lfloor k/2 \rfloor) - 4k\right) \ge \nu(x_t - 2\lfloor k/2 \rfloor) - 4k^2 \text{ if } t \in T_2. \tag{41}$$

Notice that $|Q_t| = 3k - 4\ell$ for all $t = 1, \dots, n - 1$ and, as Q'_0 has at least 3 edges, $|Q_0| \ge 2(k - 2\ell) + 3(k - \ell) + \ell \ge 3k - 4\ell$. Using these estimates and recalling (34), (35), (40), and (41), we now bound from below the total number N' of vertices appearing in all so far constructed objects.

$$\begin{aligned} N' &= \sum_{t=0}^{n-1} |Q_t| + \sum_{t=1}^{n} |V(P'_t)| \\ &\ge (3k - 4\ell)n + \sum_{t \in T_1} \nu(x_t - 2\lfloor k/2 \rfloor) + \sum_{t \in T_2} (\nu(x_t - 2\lfloor k/2 \rfloor) - 4k^2) \\ &\ge (3k - 4\ell)n + \sum_{t=1}^{n} \nu(x_t - 2\lfloor k/2 \rfloor) - 4k^3 > N, \end{aligned}$$

where the last inequality holds by (10).

Trimming

Recall that N is divisible by $k - \ell$. It is easy to check that the same is true for N'. As long as $N' > N$ we apply the following iterative procedure of trimming the paths $P'_1, \dots, P'_t$: choose a path, which currently contains the largest number of vertices of W and remove from it *precisely* $k - \ell$

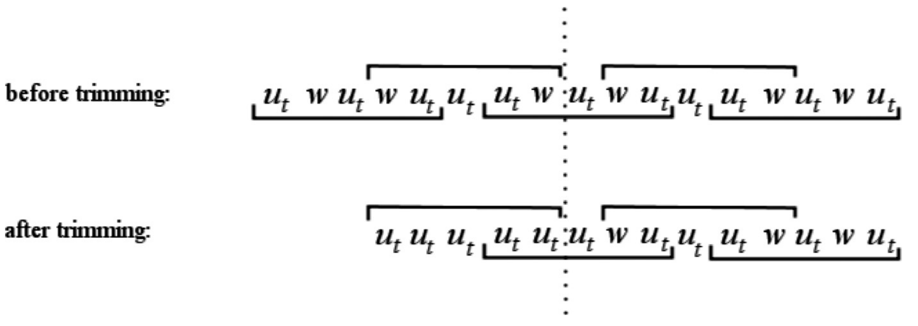

Fig. 3. Illustration of trimming for $k = 5$ and $\ell = 2$; the segment to the right of the dotted line remains unchanged, while the one to the left retains only of vertices from U_t.

leftmost vertices of W (according to the order of their appearance on the path). As, by (11)–(13), (62), (14) and Fact 22

$$\left|\bigcup_{t=1}^{n}\left(V(P_t')\cap W\right)\right| \ge N' - \sum_{t=0}^{n-1}|Q_t| - \sum_{t=1}^{n}|U_t| > N - 5kn - \sum_{t=1}^{n}|U_t|$$
$$= \sum_{t=n+1}^{2n}|U_t| - 5kn \ge n\cdot\min b_t - 5kn \ge (4k^4-5k)n, \tag{42}$$

a path with at least $k-\ell$ vertices of W exists (as long as $N' > N$). It is easy to see that, treating the remaining vertices of the truncated path as consecutive, we obtain a new, shorter (by $k-\ell$) path such that each of its edges still has at least $k-\ell+1$ vertices of $U_t \setminus Q$, see Fig. 3. Indeed, the edges to the right of the rightmost removed element (dotted line in Fig. 3) remain the same as before trimming (due to the fact that we have removed exactly $(k-\ell)$ leftmost vertices of W), while those to the left have now all vertices in $U_t \setminus Q$. For the remaining edge (the one with vertices to the left and to the right) we argue similarly. Its part to the right remains unchanged (and so has the same number of vertices from $U_t \setminus Q$ as before trimming), while the part to the left has now all vertices in $U_t \setminus Q$ (at least as many as before trimming).

We conclude the procedure when the current number of vertices in all the paths and sequences $Q_0, \dots, Q_{n-1}$ (which remain untouched) reaches N. Let the resulting paths be denoted by $P_1'', \dots, P_n''$.

Furthermore, note that by (40), (41), (47) and (59), at the beginning of the trimming we had

$$\left|V(P_t')\cap W\right| = \left|V(P_t')\setminus U_t\right| \ge \nu(x_t-k) - 4k^2 - x_t \ge \frac{k+1}{k}(x_t-k) - 4k^2 - x_t$$
$$= \frac{x_t}{k} - 4k^2 - k \ge 10k^3 - 4k^2 - k \ge 5k^3. \tag{43}$$

Since at every stage we removed vertices from a path with the largest number of vertices in W, by (42) and (43),

$$\left|V(P_t'')\cap W\right| \ge \min\{5k^3, 4k^4 - 5k - (k-\ell)\} = 5k^3. \tag{44}$$

Eradicating

We still have to eradicate the remaining vertices of W, that is, to replace them by the vertices of $\bigcup_{s=n+1}^{2n} U_s$. While doing so, we will also prepare the structure of the paths for the final concatenation into a hamiltonian (ℓ, k)-cycle. In fact, this preparation will mostly affect only the first edge, call it f_t'', of P_t'' for $t \in T_1$.

Preparation: We first change the order of the first k vertices of P''_t, so that the vertices on positions $\ell+1, \ell+2, \dots k$ are all from U_t. This is possible because f''_t (as well as every other edge of P''_t) contains at least $k-\ell+1$ vertices from U_t. Note that this operation may also affect the second edge of P''_t, but it will still have at least $k-\ell+1$ vertices from U_t. The remaining edges of P''_t, as disjoint from f''_t, remain unchanged. Let us call the resulting path P'''_t and its first edge f'''_t. Focusing on f'''_t, we see that among its first ℓ vertices at least one is from U_t (because f'''_t has at least $k-\ell+1$ vertices from U_t). Now, if there are more than one vertices like this, we swap all but one of them with arbitrary vertices of $W \cap \left(P'''_t \setminus f'''_t\right)$ (note that by (44) there are enough vertices of W in P'''_t to do this). After this operation the number of vertices from U_t in every edge (but f'''_t) can only increase, so still each edge has at least $k-\ell+1$ vertices from U_t.

Finally, if necessary, we move the unique vertex of U_t among the first ℓ vertices to the ℓ-th position and, if it belongs to B_t, we exchange it with a vertex of A_t (which also belongs to P'''_t). Such a vertex exists, since, by Fact 22(i), out of all vertices of A_t, precisely $2\lfloor k/2 \rfloor$ were used by Q, while the remaining ℓ are sitting somewhere on the path P'''_t. In summary, after these changes we obtain a new path P''''_t such that, for each $t \in T_1$, the structure of its first edge is

$$f''''_t = \underbrace{w, \dots, w}_{\ell-1}, a_t, \underbrace{u_t \dots, u_t}_{k-\ell}. \tag{45}$$

Replacement: Finally, to obtain the desired paths $P_t \in H_2$, we replace the vertices of W in $\bigcup_{t=1}^{n} V(P''''_t)$ by the vertices of $\bigcup_{s=n+1}^{2n} U_s$ in the following order. First, for each $t \in T_1$, we replace the $\ell - 1$ vertices of W at the left end of f''''_t by vertices from A_{n+t}. This is possible, since by (11) and Fact 22, there are at least $k-3 \geq \ell-1$ vertices of A_{n+t} unused so far. As a result, the first edge of each path P_t, $t \in T_1$, by (45), takes the form

$$f_t = \underbrace{a_{n+t}, \dots, a_{n+t}}_{\ell-1}, a_t, \underbrace{u_t \dots, u_t}_{k-\ell}. \tag{46}$$

The remaining vertices of W in $\bigcup_{t=1}^{n} V(P''''_t)$ are replaced arbitrarily.

4.6. Construction of the hamiltonian cycle C

We will show that the following sequence

$$C = Q_0, P_1, Q_1, P_2, Q_2, P_3, \dots, Q_{n-1}, P_n.$$

spans a hamiltonian (ℓ, k)-cycle in $H_1 \cup H_2 + e$. Recall that for each $t \in [n]$, $P_t \subseteq H_2$. Also, each sequence Q_t, $t \in [0, n-1]$, consists of a core path ($Q'_0 \subseteq H_1 \cup H_2 + e$ for $t = 0$ and just one edge $e_t \in H_1$ for $t \in [n-1]$) and two "loose ends" of $k - 2\ell$ vertices each. Thus, there are exactly $2n$ edges of C which are not contained in $Q_0 \cup P_1 \cup \cdots \cup Q_{n-1} \cup P_n$ and require a proof that they also belong to $H_1 \cup H_2$. Each of these new edges shares exactly $k-\ell$ vertices with a Q_t and ℓ vertices with either P_t (P_n for $t = 0$) or P_{t+1}, $t = 0, \dots, n-1$. Let us denote them by g^L_t and g^R_t, respectively (see Figs. 4 and 5). For convenience, we set $P_0 = P_n$.

Let us first focus on g^L_t, $t \in [0, n-1]$. By the construction of Q_t (see (30)–(33) for $t = 0$ and (39) for $t \geq 1$), we have $g^L_t \cap Q_t \subset U_t$, so $|g^L_t \cap Q_t \cap U_t| = k - \ell$. Further, as $P_t \subset H_2$, among its last ℓ vertices there must be at least one from U_t. Since $|g^L_t \cap V(P_t)| = \ell$, it altogether yields that $g^L_t \in H_2$. In the same way one can prove that $g^R_t \in H_2$ for all t such that $t+1 \in T_2$ (see Fig. 4).

Finally, consider g^R_t with $t+1 \in T_1$, (see Fig. 5). By (39) and (46) we have $\{t+1, n+t+1\} \in tr(g^R_t)$, $|g^R_t \cap A_{t+1+n}| = k - \ell - 1$ and $|g^R_t \cap A_{t+1}| = \ell + 1$. Hence, $g^R_t \in H^2_1$.

5. Concluding remarks

After fixing an inaccuracy in the first version of our proof, it turned out, quite disappointedly, that Theorem 2, and thus Corollary 3, does not cover the case $\ell = \lfloor k/2 \rfloor = (k-1)/2$ for odd k. However, a few little changes in the proof can close this gap. In order to confirm Conjecture 1 for

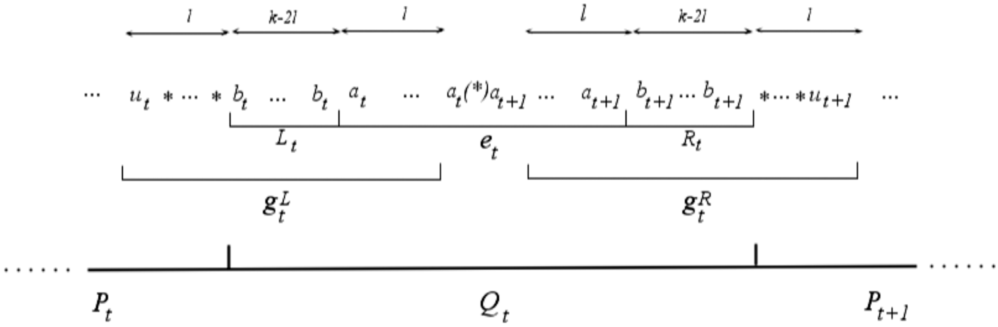

Fig. 4. Construction of C, $t+1 \in T_2$.

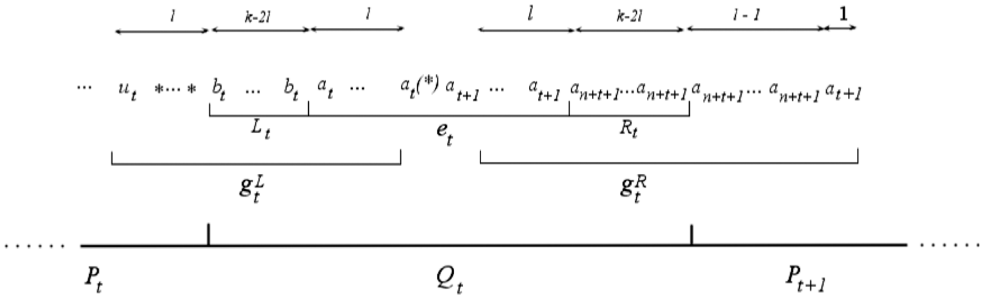

Fig. 5. Construction of C, $t+1 \in T_1$.

$\ell = (k-1)/2$, one has to prove Lemmas 10 and 11 for

$$2 \le p = \ell = (k-1)/2,$$

which together will imply a corresponding version of Theorem 2 for $p = \ell = (k-1)/2$, and thus Conjecture 1 for $\ell = (k-1)/2$.

The change in the proof boils down to replacing $2\lfloor k/2 \rfloor + \ell$ with $2\lfloor k/2 \rfloor + \ell - 1$ in (11) and, accordingly, $x_i - 2\lfloor k/2 \rfloor - \ell$ with $x_i - 2\lfloor k/2 \rfloor - \ell + 1$ in (12). Notice that, for each i, $|U_i| = |A_i| + |B_i|$ stays unchanged. As a result, (20) and (21) remain true, since now $|A_j| \le 3\lfloor k/2 \rfloor - 1$. Moreover, although inequality (24) is relaxed to

$$\left|\left(V(P_i) \cap A_{j_i}\right) \setminus (e_1 \cup \cdots \cup e_m)\right| \ge \ell,$$

it still implies that $deg_{Tr(M)}(j_i) \le 1$, because $\left|A_{j_i}\right| \le 2\lfloor k/2 \rfloor + \ell - 1$. This saves Claims 16 and 17, while all estimates of the length of C remain intact (they rely mainly on the cardinalities of U_t which have not changed). Thus, the proof of Lemma 10 is retained.

In order to modify the proof of Lemma 11, in Section 4.2 one has to choose i and j according to whether $\rho(C_1) \le \ell - 1$ or $\rho(C_1) \ge \ell$, instead of $\rho(C_1) \le \ell$ or $\rho(C_1) \ge \ell + 1$ (and the same for $\rho(C_2)$). This does not affect the structural properties of the bridge Q_0, as consecutive edges intersect in ℓ vertices only, but at the same time strengthens Fact 20 to $|e \cap U_t| \le \ell - 1$. This, in turn, allows one to replace the middle part of Fact 21 by $|Q_0 \cap A_t| \le \ell - 1$, compensating for the decrease of $|A_t|$.

Indeed, since all bridges $Q_1, \dots, Q_{n-1}$, defined in (39), use together at most $2\lfloor k/2 \rfloor$ vertices from each set A_t, $t = 2, \dots, n-1$, this part of Fact 21 implies that there are sufficiently many vertices in the sets A_t, $t = 2, \dots, n-1$, to construct all bridges (including Q_0). On the other hand, for each $t \in \{1, n\}$, the bridges $Q_1, \dots, Q_{n-1}$ require only at most $\lfloor k/2 \rfloor$ vertices from A_t. Hence, by the first line of Fact 21 (with $p = \ell = (k-1)/2$), we have

$$k - p + \lfloor k/2 \rfloor = k \le 3(k-1)/2 - 1 = |A_t|,$$

since $k \ge 5$. Consequently, the construction of all bridges can be completed. As the remainder of the proof of Lemma 11 does not involve the (modified) cardinalities of the sets A_t, the construction of the Hamiltonian cycle C can be finalized basically in the same way as presented in Sections 4.5 and 4.6.

Let us summarize that, owing to the above extension, Conjecture 1 is now confirmed for $\ell = 1$, all $(k-1)/3 \le \ell \le k/2$, and all $\ell \ge 0.8k$. We believe that the two missing ranges of ℓ will require some new ideas.

Acknowledgments

We would like to thank both referees for several remarks and suggestions leading to a great improvement of the exposition of the paper. We are especially indebted to referee X who found an inaccuracy in an earlier version of the proof (c.f. Concluding Remarks).

Appendix. Properties of function ν

In [7] we proved the following simple facts.

Proposition 23 ([7]). *Function ν has the following properties.*

(a) For every $x \ge (k-3)(k-1)$, $\quad \nu(x) \ge x + \left\lfloor \frac{x}{k-1} \right\rfloor + 3 - k$.
(b) For every $x \ge k-2$, $\quad \nu(x) \le kx$.
(c) For all $x \ge 2$, $\quad \nu(x-1) \ge \nu(x) - k$.

We will now note three consequences of the above proposition. For $x \ge k^3$ it follows from Proposition 23(a) that

$$x \le \frac{k-1}{k}\nu(x) + \frac{(k-1)(k-2)}{k} \le \frac{k}{k+1}\nu(x) \le \nu(x). \tag{47}$$

Indeed, after dropping the floor in (a), we get the first inequality above, while the second inequality is equivalent to $\nu(x) \ge (k+1)(k-1)(k-2)$, which is true by the assumption on x. Moreover, since $\nu(x)$ equals ℓ modulo $k-\ell$, Proposition 23(c) can be strengthened to yield, for $x \ge 2$,

$$\nu(x) = \nu(x-1) \quad \text{or} \quad \nu(x) - \nu(x-1) = k - \ell. \tag{48}$$

Finally, by iterating the inequality of Proposition 23(c) t times, we have

$$\nu(x+t) \le \nu(x) + tk. \tag{49}$$

It follows directly from these definitions that

$$z \ge \nu\left(\mu(z)\right) \quad \text{and} \quad z < \nu\left(\mu^*(z)\right). \tag{50}$$

The following properties of functions ν, μ, and μ^* will turn out to be crucial in our proofs.

Proposition 24. *We have*

$$\nu\left(\mu^*(z)\right) - \nu\left(\mu(z)\right) = k - \ell, \tag{51}$$

$$\nu\left(\mu(z)\right) = \nu\left(\mu(z) - (k-2\ell)\right), \quad \textit{and} \quad \nu\left(\mu^*(z)\right) = \nu\left(\mu^*(z) + (k-2\ell)\right). \tag{52}$$

Proof. Equality (51) follows from (5) and (48). In order to deduce (52), we first determine an exact formula for function ν from which it will follow quickly. Set $\kappa = k - \ell + 1$ and $\beta = 2k - 4\ell + 2$

and notice that

$$\max\{\kappa,\beta\}=\begin{cases}\kappa & \text{if } \ell\geq\frac{k+1}{3},\\ \beta & \text{if } \ell<\frac{k+1}{3}.\end{cases}$$

Let us choose an integer x and define integers $q:=q(x,k,\ell)$ and $r:=r(x,k,\ell)$ by setting

$$x-\kappa=q\max\{\kappa,\beta\}+r,\quad \text{where}\quad 0\leq r\leq\max\{\kappa,\beta\}-1. \tag{53}$$

We claim that

$$\nu(x)=\begin{cases}q(2k-2\ell)+k & \text{if } r\leq k-2\ell\\ q(2k-2\ell)+2k-\ell & \text{if } r\geq k-2\ell+1\end{cases} \tag{54}$$

Formula (54) shows that $\nu(x)$ is a step functions which is constant on intervals (steps) of lengths, alternately, $k-2\ell+1$, and $\max\{\kappa,\beta\}-1-(k-2\ell)\geq\beta-1-(k-2\ell)=k-2\ell+1$. This, together with the definitions of μ and μ^*, implies equalities (52). Indeed, let, for instance, $x=\mu(z)$ for some z. Then $\nu(x)\leq z$ but $\nu(x+1)>z$. In view of (54) this means that in the expression (53) we have either $r=k-2\ell$ or $r=\max\{\kappa,\beta\}-1$, that is, x is at the right end of a step of ν. Thus, clearly, $\nu(x-(k-2\ell))=\nu(x)$, as required. For the second equality in (52), observe that if $x=\mu^*(z)$, then $\nu(x)>z$ but $\nu(x-1)\leq z$, so x sits at the left end of a step of ν.

In order to show (54), we will first prove an upper bound valid for all (ℓ,k)-paths P satisfying (4) and then construct a particular (ℓ,k)-path P_0 which achieves this bound.

Let P be an (ℓ,k)-path with t edges satisfying (4). Let $e_1,\ldots,e_t$ be the edges of P in the linear order underlying P. Set $s=\left\lfloor\frac{t+1}{2}\right\rfloor$. Clearly, $t\in\{2s-1,2s\}$. Further, set

$$f_i=e_{2i-1}\cup e_{2_i}\setminus e_{2i+1},\quad i=1,\ldots,s-1.$$

Since, by (4), $|e_{2i-1}\cap U|\geq\kappa$ for each $i\in\{1,\ldots,s\}$, we have $|f_i\cap U|\geq\kappa$ for each $i\in\{1,\ldots,s-1\}$, too. However, if $\ell<(k+1)/3$, then this bound can be improved. As, also, $|e_{2i}\cap U|\geq\kappa$ for each $i\in\{1,\ldots,s-1\}$, we infer that

$$|(e_{2i}\setminus(e_{2i-1}\cup e_{2i+1}))\cap U|\geq\kappa-2\ell=k-3\ell+1.$$

Therefore,

$$|f_i\cap U|\geq\beta\quad i=1,\ldots,s-1,\quad\text{and}\quad |e_{2s-1}\cap U|\geq\kappa.$$

Because $f_1,f_2,\ldots,f_{s-1},e_{2s-1}$ are pairwise disjoint, this implies, in view of (53), that $s-1\leq q$. Also by (4), if $t=2$ s, then

$$|(e_t\setminus e_{2s-1})\cap U|\geq\kappa-\ell=k-2\ell+1.$$

Thus, if $r\leq k-2\ell$, then $t=2s-1$ and

$$|V(P)|=\sum_{i=1}^{s-1}|f_i|+|e_{2s-1}|=(s-1)(2k-2\ell)+k\leq q(2k-2\ell)+k.$$

Otherwise, $t\leq 2s$ and

$$|V(P)|=\sum_{i=1}^{s-1}|f_i|+|e_{2s-1}\cup e_{2s}|=(s-1)(2k-2\ell)+2k-\ell$$
$$\leq q(2k-2\ell)+2k-\ell.$$

To show equality, let us construct P_0 satisfying (4) which achieves this bound. We will represent P_0 as a binary sequence Q over the alphabet $\{u,w\}$, where each vertex of U is represented by u and each vertex of $V(P_0)\cap W$ is represented by w (and the edges of P_0 follow the sequence Q according to the definition of an (ℓ,k)-path).

Assume first that $\ell\geq\frac{k+1}{3}$. Sequence Q consists of q identical blocks plus another block at the end (see diagram (55) below). Each block begins with a u-run of length $\kappa-\ell$, followed by a w-run of length $\ell-1$, followed by a u-run of length ℓ, followed by a w-run of length $k-2\ell$. The final block begins with the same runs as all previous blocks, that is, a u-run of length $\kappa-\ell$, followed by a w-run of length $\ell-1$, followed by a u-run of length ℓ. If $r \le k-2\ell$, then this is it, except that we arbitrarily convert r symbols w to u. If $r \ge k-2\ell+1$, we add a u-run of length r followed by a w-run of length $k-\ell-r$, creating one more edge. In this case there is no need for any final alteration.

$$\begin{gathered}\overbrace{\underbrace{u,\dots,u}_{\kappa-\ell}\underbrace{w,\dots,w,}_{\ell-1}\underbrace{u,\dots,u}_{\ell}}^{e_1}\underbrace{w,\dots,w}_{k-2\ell}\overbrace{\underbrace{u,\dots,u}_{\kappa-\ell}\underbrace{w,\dots,w,}_{\ell-1}\underbrace{u,\dots,u}_{\ell}}^{e_3}\underbrace{w,\dots,w}_{k-2\ell}\\ \dots\\ \overbrace{\underbrace{u,\dots,u}_{\kappa-\ell}\underbrace{w,\dots,w,}_{\ell-1}\underbrace{u,\dots,u}_{\ell}}^{e_{2q-1}}\underbrace{w,\dots,w}_{k-2\ell}\overbrace{\underbrace{u,\dots,u}_{\kappa-\ell}\underbrace{w,\dots,w,}_{\ell-1}\underbrace{u,\dots,u}_{\ell}}^{e_{2q+1}}(\underbrace{u,\dots,u,}_{r}\ \underbrace{w,\dots,w}_{k-\ell-r})\end{gathered} \tag{55}$$

It is easy to check that P_0 satisfies (4). Indeed, the number of symbols u equals $q\kappa+\kappa+r=x$ which agrees with (53). Moreover, every edge of P_0 covers at least κ symbols u. This is clearly seen on diagram (55) for edges e_{2i+1}, $i=1,\dots,q$. However, since $\ell \ge k-\ell$, every edge e_{2i}, $i=1,\dots,q$, also contains at least $\kappa-\ell+\ell=\kappa$ symbols u. And the last edge, e_{2q+2}, if present, contains at least $\ell+r \ge \ell+(k-2\ell+1)=\tau$ symbols u too. (We write "at least" as we do not count possible converts from w to u.) Finally, as desired (cf. (54)),

$$|V(P_0)| = \begin{cases} q(k+k-2\ell)+k = q(2k-2\ell)+k & \text{if } r \le k-2\ell \\ q(k+k-2\ell)+k+(k-\ell) = q(2k-2\ell)+2k-\ell & \text{if } r \ge k-2\ell+1. \end{cases} \tag{56}$$

For $\ell < (k+1)/3$ we modify the above construction by replacing each w-run of length $k-2\ell$ by a u-run of length $k-3\ell+1$ followed by a w-run of length $\ell-1$. Again, it is easy to check that both, (4) and (56), hold. Indeed, the total number of symbols u is $q(\kappa+k-3\ell+1)+\kappa+r = q\beta+\kappa+r$ which, again, agrees with (53). Moreover, each edge of P_0 covers at least κ symbols u. Again, this is clear for odd-index edges, while for even indices notice that, this time, $\ell < \kappa-\ell$, so these edges contain each at least $\ell+(k-3\ell+1)+\ell=\tau$ symbols u. Finally, the above modification of our construction does not change the total number of vertices in P_0, so $|V(P_0)|$ is the same as in (56). □

By (52) in Proposition 24 and the definitions of x and x^* above,

$$\nu(x-2\lfloor k/2\rfloor) = \nu(\mu(z)) = \nu(\mu(z)-(k-2\ell)) = \nu\left(x-2\lfloor k/2\rfloor-(k-2\ell)\right) \tag{57}$$

and

$$\nu(x^*-2\lfloor k/2\rfloor) = \nu(\mu^*(z)+(k-2\ell)) = \nu(\mu^*(z)) = \nu\left(x^*-2\lfloor k/2\rfloor-(k-2\ell)\right). \tag{58}$$

Also, by Proposition 23(b), the monotonicity of ν, (51), (50), the definition of z in (9), and (8),

$$\begin{aligned} x &\ge \frac{\nu(x)}{k} \ge \frac{\nu(\mu(z))}{k} = \frac{\nu(\mu^*(z))-(k-\ell)}{k} > \frac{z-k}{k} \\ &\ge \frac{N}{kn}+\frac{4k^2}{n}-4 \ge 11k^4-4 \ge 10k^4. \end{aligned} \tag{59}$$

In particular, $x-2\lfloor k/2\rfloor \ge k^3$, which justifies several future applications of (47).

On the other hand, by (47),(9), (49), (50), and (8),

$$x \le \nu(x) \le \nu(\mu(z)+k) \le \nu(\mu(z))+k^2 \le z+k^2 \le \frac{N+4k^3}{n}+k^2 \le 12k^5. \tag{60}$$

Proposition 25. *There exist $x_i \in \{x, x^*\}$, $i = 1, \dots, n$, such that*

$$nz < \sum_{i=1}^{n} \nu(x_i - 2\lfloor k/2 \rfloor) \le nz + k - \ell. \tag{61}$$

Proof. Set $y := \nu(x - 2\lfloor k/2 \rfloor)$ and $y^* = \nu(x^* - 2\lfloor k/2 \rfloor)$. By (9), (57), and (58), $y = \nu(\mu(z))$ and $y^* = \nu(\mu^*(z))$. Thus, by (51), $y^* - y = k - \ell$, and, by (50), $y \le z$ while $y^* > z$. We are going to show by induction on $m = 1, \dots, n$ that there exists a choice of $x_i \in \{x, x^*\}$, $i = 1, \dots, m$, such that (61) is satisfied with n replaced by m. Indeed, let $x_1 = x^*$, then $z < y^* = y + (k - \ell) \le z + (k - \ell)$. Fix $2 \le m \le n$ and assume the statement is true for $m - 1$. Set $\Sigma := \sum_{i=1}^{m-1} \nu(x_i - 2\lfloor k/2 \rfloor)$. Then

$$mz < \Sigma + y^* \le mz + 2(k - \ell), \quad \text{while} \quad mz - (k - \ell) < \Sigma + y \le mz + (k - \ell).$$

Since $(\Sigma + y^*) - (\Sigma + y) = k - \ell$, we have either $\Sigma + y^* \le mz + (k - \ell)$ or $mz < \Sigma + y$, which completes the proof. □

Proof of Proposition 5. The R-H-S of (10) is the L-H-S of (61). On the other hand, by the R-H-S of (61), (47), and (59),

$$\begin{aligned}
\sum_{i \in I} \nu(x_i - 2\lfloor k/2 \rfloor) &\le \sum_{i=1}^{n} \nu(x_i - 2\lfloor k/2 \rfloor) - \min_{1 \le i \le n} \nu(x_i - 2\lfloor k/2 \rfloor) \\
&\le N + 4k^3 - (3k - 4\ell)n + k - \nu(x - 2\lfloor k/2 \rfloor) \\
&\le N + 4k^3 - (3k - 4\ell)n + 2k - x \le N - (3k - 4\ell)n - 8k^4,
\end{aligned}$$

which is the L-H-S of (10). □

Indeed, by (11), (12), (9), (47), (61), and (8)

$$\begin{aligned}
\sum_{i=1}^{2n} |A_i| + \sum_{i=1}^{n} |B_i| &= \sum_{i=1}^{n} x_i + n\,(2k - 2\ell - 3) = \sum_{i=1}^{n} (x_i - 2\lfloor k/2 \rfloor) + n\,(2\lfloor k/2 \rfloor + 2k - 2\ell - 3) \\
&\le \frac{k}{k+1} \sum_{i=1}^{n} \nu(x_i - 2\lfloor k/2 \rfloor) + 3kn < \frac{k}{k+1} \left(N + 4k^3 - (3k - 4\ell)n + 2k\right) + 3kn \\
&< N - \frac{N}{k+1} + \frac{k^2}{k+1}\left(4k^2 - 3n + 2\right) + 5kn < N - \left(\frac{N}{k+1} - 5kn\right) < N - 4k^4 n.
\end{aligned}$$

Thus, for each $i = n + 1, \dots, 2n$, we have

$$b_i \ge \left\lfloor \frac{1}{n} \sum_{j=n+1}^{2n} b_j \right\rfloor \ge 4k^4. \tag{62}$$

while, trivially,

$$b_i \le \left\lceil \frac{1}{n} \sum_{j=n+1}^{2n} b_j \right\rceil \le N/n + 1 \le 12k^5, \tag{63}$$

where the last inequality follows by (8). □

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