Approximate Counting of Matchings in $(3,3)$-Hypergraphs*

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Abstract. We design a fully polynomial time approximation scheme (FPTAS) for counting the number of matchings (packings) in arbitrary 3-uniform hypergraphs of maximum degree three, referred to as $(3,3)$-hypergraphs. It is the first polynomial time approximation scheme for that problem, which includes also, as a special case, the 3D Matching counting problem for 3-partite $(3,3)$-hypergraphs. The proof technique of this paper uses the general correlation decay technique and a new combinatorial analysis of the underlying structures of the intersection graphs. The proof method could be also of independent interest.

1 Introduction

The computational status of approximate counting of matchings in hypergraphs has been open for some time now, contrary to the existence of polynomial time approximation schemes for graphs. The matching (packing) counting problems in hypergraphs occur naturally in the higher dimensional free energy problems, like in the monomer-trimer systems discussed, e.g, by Heilmann [10]. The corresponding optimization versions of hypergraph matching problem relate also to various allocations problems.

This paper aims at shedding some light on the approximation complexity of that problem in 3-uniform hypergraphs of maximum vertex degree three (called $(3,3)$-hypergraphs or $(3,3)$-graphs for short). This class of hypergraphs includes

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also so-called 3D hypergraphs, that is, (3,3)-graphs that are 3-partite. In [12], based on a generalization of the canonical path method of Jerrum and Sinclair [11], we established a fully polynomial time randomized approximation scheme (FPRAS) for counting matchings in the classes of k-uniform hypergraphs without structures called 3-combs. However, the status of the problem in arbitrary $(3, 3)$-graphs was left wide open among with other general problems for 3-, 4- and 5-uniform hypergraphs (for $k \geq 6$ it is known to be hard, see Sec. 2). In particular, the existence of an FPRAS for counting matchings in $(3, 3)$-graphs was unknown.

In this paper we design the first fully polynomial time approximation scheme (FPTAS) for arbitrary $(3, 3)$-graphs. The method of solution depends on the general correlation decay technique and some new structural analysis of underlying intersections graphs based on an extension of the classical claw-freeness notion. The proof method used in the analysis of our algorithm could be also of independent interest.

The paper is organized as follows. Section 2 contains some basic notions and preparatory discussions. In Sec. 3 we formulate our main results and provide the proofs. Finally, Sec. 4 is devoted to the summary and an outlook for future research.

2 Preliminaries

A hypergraph $H = (V, E)$ is a finite set of vertices V together with a family E of distinct, nonempty subsets of vertices called edges. In this paper we consider *k-uniform hypergraphs* (called further *k-graphs*) in which, for a fixed $k \geq 2$, each edge is of size k. *A matching* in a hypergraph is a set (possibly empty) of disjoint edges.

Counting matchings is a #P-complete problem already for graphs ($k = 2$) as proved by Valiant [17]. In view of this hardness barrier, researchers turned to approximate counting, which initially has been accomplished via probabilistic techniques.

Given a function C and a random variable Y (defined on some probability space), and given two real numbers $\epsilon, \delta > 0$, we say that Y is an (ϵ, δ)-*approximation* of C if the probability $\mathbb{P}\left(|Y(x) - C(x)| \geq \epsilon C(x)\right) \leq \delta$. A *fully polynomial randomized approximation scheme* (FPRAS) for a function f on $\{0, 1\}^*$ is a randomized algorithm which, for every triple (ϵ, δ, x), with $\epsilon > 0$, $\delta > 0$, and $x \in \{0, 1\}^*$, returns an (ϵ, δ)-approximation Y of $f(x)$ and runs in time polynomial in $1/\epsilon$, $\log(1/\delta)$, and $|x|$.

In this paper we investigate the problem of counting the number of matchings in hypergraphs and try to determine the status of this problem for k-graphs with bounded degrees.

Let $deg_H(v)$ be the degree of vertex v in a hypergraph H, that is, the number of edges of H containing v. We denote by $\Delta(H)$ the maximum of $deg_H(v)$ over all v in H. We call a k-graph H a *(k, r)-graph* if $\Delta(H) \leq r$. Let $\#M(k, r)$ be the problem of counting the number of matchings in (k, r)-graphs.

Our inspiration comes from new results (both positive and negative) that emerged for approximate counting of the number of independent sets in graphs with bounded degree and shed some light on the problem $\#M(k,r)$.

Let $\#IS(d)$ [$\#IS(\leq d)$] be the problem of counting the number of all independent sets in d-regular graphs [graphs of maximum degree bounded by d, that is, $(2,d)$-graphs]. Luby and Vigoda [14] established an FPRAS for $\#IS(\leq 4)$. This was complemented later by the approximation hardness results for the higher degree instances by Dyer, Frieze and Jerrum [7]. The subsequent progress has coincided with the revival of a deterministic technique – the spatial correlation decay method – based on early papers of Dobrushin [5] and Kelly [13]. It resulted in constructing deterministic approximation schemes for counting independent sets in several classes of graphs with degree (and other) restrictions, as well as for counting matchings in graphs of bounded degree.

Definition 1. *A fully polynomial time approximation scheme (***FPTAS***) for a function f on $\{0,1\}^*$ is a deterministic algorithm which for every pair (ϵ, x) with $\epsilon > 0$, and $x \in \{0,1\}^*$, returns a number $y(x)$ such that*

$$|y(x) - f(x)| \leq \epsilon f(x),$$

and runs in time polynomial in $1/\epsilon$, and $|x|$.

In 2007 Weitz [18] found an FPTAS for $\#IS(\leq 5)$, while, more recently, Sly [15] and Sly and Sun [16] complemented Weitz's result by proving the approximation hardness for $\#IS(6)$, that is, proving that unless NP=RP, there exists no FPRAS (and thus, no FPTAS) for $\#IS(6)$. By applying two reductions: from $\#IS(6)$ to $\#M(6,2)$ (taking the dual hypergraph of a 6-regular graph), and from $\#M(k,2)$ to $\#IS(k)$ (taking the intersection graph of a $(k,2)$-graph) for $k = 3,4,5$, we conclude that

(i) (unless NP=RP) there exists no FPRAS for $\#M(6,2)$;
(ii) there is an FPTAS for $\#M(k,2)$ with $k \in \{3,4,5\}$.

Note that the first reduction results, in fact, in a *linear* $(6,2)$-graph, so the class of hypergraphs in question is even narrower. (A hypergraph is called *linear* when no two edges share more than one vertex.) On the other hand, by the same kind of reduction it follows from a result of Greenhill [9] that *exact* counting of matchings is #P-complete already in the class of linear $(3,2)$-graphs.

Facts (i) and (ii) above imply that the only interesting cases for the positive results are those for (k,r)-graphs with $k = 3,4,5$ and $r \geq 3$, and thus, the smallest one among them is that of $(3,3)$-graphs. Our main result establishes an FPTAS for counting the number of matchings in this class of hypergraphs.

3 Main Result and the Proof

The following theorem is the main result of this paper.

Theorem 2. *The algorithm CountMatchings given in Section 3.2 provides an FPTAS for* $\#M(3,3)$ *and runs in time* $O\left(n^2(n/\epsilon)^{\log_{50/49} 144}\right)$.

The intersection graph of a hypergraph H is the graph $G = L(H)$ with vertex set $V(G) = E(H)$ and edge set $E(G)$ consisting of all intersecting pairs of edges of H. When H is a graph, the intersection graph $L(H)$ is called *the line graph* of H. Graphs which are line graphs of some graphs are characterized by 9 forbidden induced subgraphs [3], one of which is the *claw*, an induced copy of $K_{1,3}$. There is no similar characterization for intersection graphs of k-graphs. Still, it is easy to observe that for any k-graph H, its intersection graph $L(H)$ does not contain an induced copy of $K_{1,k+1}$. We shall call such graphs $(k+1)$-*claw-free*.

Our proof of Thm. 2 begins with an obvious observation that counting the number of matchings in a hypergraph H is equivalent to counting the number of independent sets in the intersection graph $G = L(H)$. More precisely, let $Z_M(H)$ be the number of matchings in a hypergraph H and, for a graph G, let $Z_I(G)$ be the number of independent sets in G. (Note that both quantities count the empty set.) Then $Z_M(H) = Z_I(L(H))$.

To approximately count the number of independent sets in a graph $G = L(H)$ for a $(3,3)$-graph H, we apply some of the ideas from [2] (the preliminary version of this paper appeared in [1]) and [8]. In [2] two new instances of FPTAS were constructed, both based on the spatial correlation decay method. First, for $\#M(2,r)$ with any given r. Then, still in [2], the authors refined their approach to yield an FPTAS for counting independent sets in claw-free graphs of bounded clique number which contain so called *simplicial cliques*. The last restriction has been removed by an ingenious observation in [8].

Papers [2, 8] inspired us to seek adequate methods for $(3,3)$-graphs. Indeed, for every $(3,3)$-graph H its intersection graph $G = L(H)$ is 4-claw-free and has $\Delta(G) \leq 6$. This turned out to be the right approach, as we deduced our Thm. 2 from a technical lemma (Lem. 3 below) which constructs an FPTAS for the number of independent sets in $K_{1,4}$-free graphs G with $\Delta(G) \leq 6$ and an additional property stemming from their being intersection graphs of $(3,3)$-graphs.

3.1 Proof of Theorem 2 – Sketch and Preliminaries

We deduce Thm. 2 from a technical lemma. The assumptions of this lemma reflect some properties of the intersection graphs of $(3,3)$-graphs.

Lemma 3. *There exists an FPTAS for the problem of counting independent sets in every 4-claw-free graph with maximum degree at most 6 and such that the neighborhood of every vertex of degree* $d \geq 5$ *induces a subgraph that spans a matching of size* $\lfloor d/2 \rfloor$.

Proof (of Thm. 2). Given a $(3,3)$-graph H, consider its intersection graph G. Then G is 4-claw-free, has maximum degree at most 6 and every vertex neighborhood of size $d \geq 5$ must span in G a matching of size $\lfloor d/2 \rfloor$. This means that

Lem. 3 applies to G and there is an FPTAS for counting independent sets of G which is the same as counting matchings in H. □

It remains to prove Lem. 3. We begin with underlining some properties of 4-claw-free graphs which are relevant for our method. First, we introduce the notion of a *simplicial 2-clique* which is a generalization of a simplicial clique introduced in [4] and utilized in [2]. Throughout we assume notation $A \setminus B$ for set differences and, for $A \subset V(G)$, we write $G - A$ for the graph operation of deleting from G all vertices belonging to A. In other words, $G-A = G[V(G)\setminus A]$. Also, for any graph G, we use $\delta(G)$ to denote its minimum vertex degree and $\alpha(G)$ for the size of the largest independent set in G.

Definition 4. *A set $K \subseteq V(G)$ is a* 2-clique *if $\alpha(G[K]) \leq 2$. A 2-clique is* simplicial *if for every $v \in K$, $N_G(v) \setminus K$ is a 2-clique in $G - K$.*

For us a crucial property of simplicial 2-cliques is that if G is a connected graph containing a nonempty simplicial 2-clique K then it is easy to find another simplicial 2-clique in the induced subgraph $G - K$, and consequently, the whole vertex set of G can be partitioned into blocks which are simplicial 2-cliques in suitable nested sequence of induced subgraphs of G (see Claim 8).

However, in the proof of Lem. 3 we shall use a special class of 2-cliques.

Definition 5. *A 2-clique K in a graph G is called* a block *if $|K| \leq 4$ and $\delta(G[K]) \geq 1$ whenever $|K| = 4$. A block K is* simplicial *if for every $v \in K$ the set $N_G(v) \setminus K$ is a block in $G - K$.*

Next, we state a trivial but useful observation which follows straight from the above definition. (We consider the empty set as a block too.)

Fact 6. *If K is a (simplicial) block in G then for every $V' \subseteq V(G)$ the set $K \cap V'$ is a (simplicial) block in the induced subgraph $G[V']$ of G.*

Let a graph G satisfy the assumptions of Lem. 3. The next claim provides a vital, "self-reproducing" property of blocks in G.

Claim 7. *If K is a simplicial block in G, then for every $v \in K$ the set $N_G(v) \setminus K$ is a simplicial block in $G - K$.*

Proof. Set $K_v := N_G(v) \setminus K$ for convenience. By definition of K, K_v is a block. It remains to show that K_v is simplicial. Let $u \in K_v$ and let $K_u = N_G(u) \setminus (K \cup N_G(v))$. Suppose there is an independent set I in $G[K_u]$ of size $|I| = 3$. Then u, v and the vertices of I would form an induced $K_{1,4}$ in G with u in the center. As this is a contradiction, we conclude that K_u is a 2-clique.

To show that K_u is indeed a block, note first that, by the assumptions that $\Delta(G) \leq 6$, we have $|K_u| \leq 5$. However, if $|K_u| = 5$ then v would be an isolated vertex in $G[N_G(u)]$. But $G[N_G(u)]$ spans a matching of size 3 since $\delta(u) = 6$ – a contradiction. For the same reason, if $|K_u| = 4$ then regardless of the degree of u in G (which might be 5 or 6) there can be no isolated vertex in $G[K_u]$, since $G[K_u]$ must span a matching of size 2. □

Our next claim asserts that once there is a nonempty block in G, one can find a suitable partition of $V(G)$ into sets which are blocks in a nested sequence of induced subgraphs of G defined by deleting these sets one after another.

Claim 8. *Let K be a nonempty simplicial block in G. If, in addition, G is connected then there exists a partition $V(G) = K_1 \cup \cdots \cup K_m$ such that $K_1 = K$ and for every $i = 2, \ldots, m$, K_i is a nonempty, simplicial block in $G_i := G - \bigcup_{j=1}^{i-1} K_j$.*

Proof. Suppose we have already constructed disjoint sets $K_1 \cup \cdots \cup K_s$, for some $s \geq 1$, such that $K_1 = K$, for every $i = 2, \ldots, s$, K_i is a nonempty, simplicial block in $G_i := G - \bigcup_{j=1}^{i-1} K_j$, and that $R_s := V(G) \setminus \bigcup_{i=1}^{s} K_i \neq \emptyset$. Since G is connected, there is an edge between a vertex in R_s and a vertex $v \in K_i$ for some $1 \leq i \leq s$. Since K_i is a simplicial block in G_i, by Fact 6, it is also simplicial in its subgraph $G_i[V']$, where $V' = K_i \cup R_s$, that is the subgraph of G_i obtained by deleting all vertices of $K_{i+1} \cup \cdots \cup K_s$. Now apply Claim 7 to $G_i[V']$, K_i, and v, to conclude that $N_G(v) \cap R_s$ is a simplicial block in $G_{s+1} := G - \bigcup_{i=1}^{s} K_i$. □

Let $K_1, K_2, \ldots, K_m$ be as in Claim 8. Then,

$$Z_I(G) = \frac{Z_I(G_1)}{Z_I(G_2)} \cdot \frac{Z_I(G_2)}{Z_I(G_3)} \cdot \ldots \cdot \frac{Z_I(G_i)}{Z_I(G_{i+1})} \cdot \ldots \cdot \frac{Z_I(G_m)}{Z_I(G_{m+1})}, \tag{1}$$

where $G_{m+1} = \emptyset$ and $Z_I(G_{m+1}) = 1$. Observe that for each i, $G_{i+1} = G_i - K_i$ and the reciprocal of each quotient in (1) is precisely the probability

$$\mathbb{P}_{G_i}(K_i \cap \mathbf{I} = \emptyset) = \frac{Z_I(G_i - K_i)}{Z_I(G_i)}, \tag{2}$$

where $\mathbf{I}$ is an independent set of G_i chosen uniformly at random. In view of this, the main step in building an FPTAS for $Z_I(G)$ will be to approximate the probability $\mathbb{P}_G(K_i \cap \mathbf{I} = \emptyset)$ within $1 \pm \frac{\epsilon}{n}$ (see Sec. 3.2 and Algorithm 2 therein).

But what if G is disconnected or does not contain a simplicial block to start with? First, if $G = \bigcup_{i=1}^{c} G_i$ consists of c connected components $G_1, \ldots, G_c$, then, clearly

$$Z_I(G) = \prod_{i=1}^{c} Z_I(G_i) \tag{3}$$

and the problem reduces to that for connected graphs.

As for the second obstacle, Fadnavis [8] proposed a very clever observation to cope with it. Let G be a connected graph satisfying the assumptions of Lem. 3 and let $v \in V(G)$ be such that $G - v$ is connected. By considering the fate of vertex v, we obtain the recurrence

$$Z_I(G) = Z_I(G - v) + Z_I(G^v), \tag{4}$$

where $G^v = G - N_G[v]$ and $N_G[v] = N_G(v) \cup \{v\}$. Let $G^v = \bigcup_{i=1}^{c} G_i^v$ be the partition of G^v into its connected components. For each i let $u_i \in N_G(v)$ be such that $N_G(u_i) \cap V(G_i^v) \neq \emptyset$. Owing to the connectedness of $G - v$, a vertex u_i must exist. Set $K_i = N_G(u_i) \cap V(G_i^v)$.

Claim 9. *The set K_i is a simplicial block in G_i^v.*

Proof. The proof is quite similar to that of Claim 7. We first prove that K_i is a block. Suppose there is an independent set I in $G[K_i]$ of size $|I| = 3$. Then u_i, v and the vertices of I would form an induced $K_{1,4}$ in G with u_i in the center. As this is a contradiction, we conclude that K_i is a 2-clique. To prove that K_i is, in fact, a block, notice that there is no edge between v and K_i. Thus, we cannot have $|K_i| = 5$ because then v would be an isolated vertex in $G[N(u_i)]$ – a contradiction with the assumption on G. If, however, $|K_i| = 4$ then v is the (only) isolated vertex in $G[N(u_i)]$ and, consequently, $\delta(G[K_i]) \geq 1$.

It remains to show that the block K_i is simplicial, that is, for every $w \in K_i$, the set $N_{G_i^v}(w) \setminus K_i$ is a block in $G_i^v - K_i$. This, however, can be proved mutatis mutandis as in the proof of Claim 7. □

For the first term of recurrence (4) we apply (4) recursively. In view of Claim 9, to the second term of recurrence (4) one can apply formula (3) and then each term $Z_I(G_i^v)$ can be approximated based on (1) and (2).

3.2 The Remainder of the Proof of Lemma 3

Hence, it remains to approximate $\mathbb{P}_G(K \cap \mathbf{I} = \emptyset) = \frac{Z_I(G-K)}{Z_I(G)}$ within $1 \pm \frac{\epsilon}{n}$, where K is a simplicial block in G. We set $N_v := N_G(v)$ and formulate the following recurrence relation by considering how an independent set may intersect K:

$$Z_I(G) = Z_I(G-K) + \sum_{v \in K} Z_I(G-(N_v \cup K)) + \frac{1}{2} \sum_{uv \notin G[K]} Z_I(G-(N_u \cup N_v \cup K))$$

or equivalently, after dividing sidewise by $Z_I(G-K)$,

$$\frac{Z_I(G)}{Z_I(G-K)} = 1 + \sum_{v \in K} \frac{Z_I(G-(N_v \cup K))}{Z_I(G-K)} + \frac{1}{2} \sum_{uv \notin G[K]} \frac{Z_I(G-(N_u \cup N_v \cup K))}{Z_I(G-K)}.$$

Here and throughout the inner summation ranges over all *ordered* pairs of *distinct* vertices of K such that $\{u, v\} \notin G[K]$. At this point, in view of symmetry, it seems redundant to consider ordered pairs (and consequently have the factor of $\frac{1}{2}$ in front of the sum), but we break the symmetry right now as we further observe that

$$\frac{Z_I(G-(N_u \cup N_v \cup K))}{Z_I(G-K)} = \frac{Z_I(G-(N_u \cup N_v \cup K))}{Z_I(G-(N_v \cup K))} \cdot \frac{Z_I(G-(N_v \cup K))}{Z_I(G-K)}.$$

By Claim 7, $N_v \setminus K$ is a simplicial block in $G - K$. We need to show that, similarly, $N_u \setminus (N_v \cup K)$ is a simplicial block in $G - (N_v \cup K)$.

Claim 10. *Let K be a simplicial block in G and let $u, v \in K$ be such that $u \neq v$ and $uv \notin G[K]$. Further, let $H := G - (N_G(v) \cup K)$. Then $N_H(u)$ is a simplicial block in H.*

Proof. By Claim 7, the set $N_G(u) \setminus K$ is a simplicial block in $G - K$. Apply Fact 6 to $N_G(u) \setminus K$ and $G - K$ with $V' = V(H)$. □

Let

$$\Pi_G(K) := \mathbb{P}(K \cap \mathbf{I} = \emptyset) = \frac{Z_I(G-K)}{Z_I(G)},$$

where $\mathbf{I}$ is a random independent set of G. Finally, setting $K_v := N_v \setminus K$ and $K_{uv} := N_u \setminus (N_v \cup K)$, and rewriting $G - (N_v \cup K) = G - K - K_v$, we get the recurrence for the probabilities:

$$\Pi_G^{-1}(K) = 1 + \sum_{v \in K} \Pi_{G-K}(K_v) \left(1 + \frac{1}{2} \sum_{uv \notin G[K]} \Pi_{G-K-K_v}(K_{uv}) \right).$$

This recurrence, in principle, allows one to compute $\Pi_G(K)$ exactly, but only in an exponential number of steps. Instead, we will approximate it by a function $\Phi_G(K,t)$, also defined recursively, which "mimics" $\Pi_G(K)$ but has a built-in time counter t.

Definition 11. *For every graph G, every simplicial block K in G and an integer $t \in \mathbb{Z}_+$, the function $\Phi_G(K,t)$ is defined recursively as follows: $\Phi_G(K,0) = \Phi_G(K,1) = 1$ as well as $\Phi_G(\emptyset,t) = 1$, while for $t \geq 2$ and $K \neq \emptyset$*

$$\Phi_G^{-1}(K,t) = 1 + \sum_{v \in K} \Phi_{G-K}(K_v, t-1) \left(1 + \frac{1}{2} \sum_{uv \notin G[K]} \Phi_{G-K-K_v}(K_{uv}, t-2) \right).$$

Now we are ready to state the algorithm *CountMatchings* for computing $Z_M(H)$ for any connected $(3,3)$-graph H and its subroutine *CountIS* for computing $Z_I(G)$ in a subgraph of $G = L(H)$ containing a simplicial block K.

Algorithm 1 *CountMatchings*(H,t)

1: $G := L(H)$.
2: $Z_M := 1$, $F := G$.
3: **while** $F \neq \emptyset$ **do**
4: Pick $v \in V(F)$ s.t. $F - v$ is connected.
5: $F^v := F - N_F[v]$
6: If $F^v = \emptyset$ then $Z_M = Z_M + 1$ and go to Line 3.
7: $F^v = \bigcup_{i=1}^{c} F_i^v$, where F_i^v are connected components of F^v.
8: **for** $i := 1$ to c **do**
9: Find K_i as in Claim 9
10: **end for**
11: $Z_M := Z_M + \prod_{i=1}^{c}$ *CountIS*(F_i^v, K_i, t)
12: $F := F - v$
13: **end while**
14: Return Z_M

Algorithm 2 *CountIS*(G, K, t)

Let $V(G) = \bigcup_{i=1}^{m} K_i$ be a partition of $V(G)$ as in Claim 8 with $K_1 = K$.
$Z_I := 1, F := G$
for $i = 1$ to m **do**
  $Z_I := \frac{Z_I}{\Phi_F(K_i, t)}$
  $F := F - K_i$
end for
Return Z_I

We will show that already for $t = \Theta(\log n)$, when Φ can be easily computed in polynomial time, the two functions become close to each other.

Note that both quantities, $\Pi_G(K)$ and $\Phi_G(K, t)$, fall into the interval $[\frac{1}{9}, 1]$. The lower bound is due to the fact that a block has at most 4 vertices and each of them has degree at most 2 in G^c, so that the total number of terms in the denominator is at most nine, five of them do not exceed 1, while eight of them do not exceed $\frac{1}{2}$. Our goal is to approximate $\Pi_G(K)$ by $\Phi_G(K, t)$, for a suitably chosen t, within the multiplicative factor of $1 \pm \epsilon/n$. In view of the above lower bound, it suffices to show that $|\Pi_G(K) - \Phi_G(K, t)| \leq \frac{\epsilon}{9n}$.

To achieve this goal, we will use the correlation decay technique which boils down to establishing a recursive bound on the above difference (cf. [2]). The success of this method depends on the right choice of a pair of functions g and h, with $g : [0, 1] \to \Re$, such that they are inverses of each other, that is, $g \circ h \equiv 1$. Then we define a function f_K of $|K| + 2e(G^c[K])$ variables, one for each vertex and each (ordered) non-edge of $G[K]$, as follows. Let $\mathbf{z} = (z_1, \ldots, z_{|K|}, z_{uv} : uv \notin G[K])$ be a vector of variables of that function. For ease of notation, we denote the set of all indices of the coordinates of function f_K by J, that is, we set $J := K \cup \{(u, v) : \{u, v\} \notin G[K]\}$. Then

$$f_K(\mathbf{z}) := f(\mathbf{z}) = g\left(\left\{1 + \sum_{v \in K} h(z_v)\left(1 + \frac{1}{2}\sum_{uv \notin G[K]} h(z_{uv})\right)\right\}^{-1}\right). \tag{5}$$

To understand the reason for this set-up, put $x := g(\Pi_G(K))$, $x_v := g(\Pi_{G-K}(K_v))$, $x_{uv} := g(\Pi_{G-K-K_v}(K_{uv}))$, and, correspondingly,

$$y := g(\Phi_G(K, t)) \quad y_v := g(\Phi_{G-K}(K_v, t-1)) \quad y_{uv} := g(\Phi_{G-K-K_v}(K_{uv}, t-2)).$$

Then, $f(\mathbf{x}) = x$ and $f(\mathbf{y}) = y$, and so the difference we are after can be expressed as $|x - y| = |f(\mathbf{x}) - f(\mathbf{y})|$. Thus, we are in position to apply the Mean Value Theorem to f and conclude that there exists $\alpha \in [0, 1]$ such that, setting $\mathbf{z}_\alpha = \alpha\mathbf{x} + (1 - \alpha)\mathbf{y}$,

$$|f(\mathbf{x}) - f(\mathbf{y})| = |\nabla f(\mathbf{z}_\alpha)(\mathbf{x} - \mathbf{y})| \leq |\nabla f(\mathbf{z}_\alpha)| \times \max_{\kappa \in J} |x_\kappa - y_\kappa|.$$

It remains to bound $\max_z |\nabla f(\mathbf{z})|$ from above, uniformly by a constant $\gamma < 1$. Then, after iterating at most t but at least $t/2$ times, we will arrive at a triple

(G', K', t'), where G' is an induced subgraph of G, K' is a block in G', and $t' \in \{0, 1\}$. At this point, setting $\mu_g := |g(1)| + |\max_s g(s))|$, we will obtain the ultimate bound

$$|x - y| \le \gamma^{t/2} \times |g(\Pi_{G'}(K')) - g(1)| \le \gamma^{t/2} \times \mu_g \le \frac{\epsilon}{9n},$$
$$\text{for} \qquad t \ge 2\log((9\mu_g n)/\epsilon)/\log(1/\gamma). \tag{6}$$

In [2], to estimate $|\nabla f(\mathbf{z})|$ for a similar function f, the authors chose $g(s) = \log s$ and $h(s) = e^s$. This choice, however, does not work for us. Instead, we set $g(s) = s^{1/4}$ and $h(s) = s^4$. Then, $\mu_g = 2$ and

$$|\nabla f(\mathbf{z})| \le \sum_{\kappa \in J} \left| \frac{\partial f(\mathbf{z})}{\partial z_\kappa} \right| = \frac{\sum_{v \in K} \left\{ z_v^3 + \frac{1}{2} \sum_{uv \notin G[K]} (z_v^3 z_{uv}^4 + z_v^4 z_{uv}^3) \right\}}{\left\{ 1 + \sum_{v \in K} z_v^4 \left(1 + \frac{1}{2} \sum_{uv \notin G[K]} z_{uv}^4 \right) \right\}^{5/4}}.$$

Observe that f_K depends only on the isomorphism type of $G[K]$, a graph on up to 4 vertices, with no independent set of size 3, and with no isolated vertex when $|K| = 4$. Let us call all these graphs *block graphs*. One block graph is given in Figure 1 below.

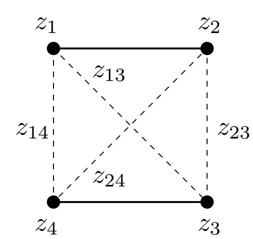

Fig. 1. The essential block graph.

In a sense we just need to consider this one block graph. Indeed, the complement of every block graph is contained in the complement of the block graph in Figure 1. Hence, it suffices to maximize $|\nabla f(\mathbf{z})|$ just for this graph. Our computational task is, therefore, to bound from above

$$F(\mathbf{z}) = \|\nabla(\mathbf{z})\|_1 = \frac{1}{4}\Big(1 + z_1^4 + z_2^4 + z_3^4 + z_4^4 + \frac{1}{2}\big(z_{14}^4(z_1^4 + z_4^4) + z_{13}^4(z_1^4 + z_3^4) + z_{23}^4(z_2^4 + z_3^4) + z_{24}^4(z_2^4 + z_4^4)\big)\Big)^{-5/4} \times$$
$$\Big(2z_1^3(2 + z_{14}^4 + z_{13}^4) + 2z_2^3(2 + z_{23}^4 + z_{24}^4) + 2z_3^3(2 + z_{13}^4 + z_{23}^4) + 2z_4^3(2 + z_{14}^4 + z_{24}^4) + 2z_{14}^3(z_1^4 + z_4^4) + 2z_{13}^3(z_1^4 + z_3^4) + 2z_{23}^3(z_2^4 + z_3^4) + 2z_{24}^3(z_2^4 + z_4^4)\Big).$$

One can show (using, e.g., *Mathematica*) that $F(\mathbf{z}) < 0.971$ for $0 \leq z_i \leq 1$ and $0 \leq z_{ij} \leq 1$. Thus, we have (6) with $\mu_g = 2$ and, say, $\gamma = 0.98 = \frac{49}{50}$. Summarizing, the running time of computing $\Phi_G(K,t)$ in Step 4 of Algorithm 2 is 12^t since there at most 12 expressions to compute in each step of the recurrence relation (see Def. 11). Also, *CountIS* takes at most $|V(F_i^v)|12^t$ steps and hence, Line 11 of *CountMatchings* takes $n12^t$ steps and is invoked at most n times. Consequently, with $t = 2\lceil \log((18n)/\epsilon)/\log(50/49)\rceil$ we get the running time of our algorithm of order $O\left(n^2(n/\epsilon)^{\log_{50/49}144}\right)$.

Remark 12. With basically the same proof we can construct an FPTAS for calculating the partition function $Z_M(H,\lambda) = \sum_M \lambda^{|M|}$, where the sum runs over all matchings in H, for any constant $\lambda \in (0, 1.077]$. The λ factor will appear in front of each summation in (5), which one can neutralize by setting $h(s) = \frac{s^4}{\lambda}$ and $g(s) = (\lambda s)^{1/4}$.

4 Summary, Discussion, and Further Research

The main result of this paper (Thm. 2) establishes an FPTAS for the problem $\#M(3,3)$ of counting the number of matchings in a $(3,3)$-graph. A reformulation of Thm. 2 in terms of graphs yields an FPTAS for the problem of counting independent sets in every graph which is the intersection graph of a $(3,3)$-graph. As mentioned earlier, every intersection graph of a $(3,3)$-graph is 4-claw-free. Moreover, its maximum degree is at most six. We wonder if there exists an FPTAS for the problem of counting independent sets in every 4-claw-free graph with maximum degree at most 6. Lemma 3 falls short of proving that. The missing part is due to our inability to repeat the above estimates for 2-cliques of size five.

In an earlier paper [12] three of the authors have found an FPRAS for the number of matchings in k-graphs without 3-combs. As their intersection graphs are claw-free, it follows from the above mentioned result on independent sets in [2, 8] that there is also an FPTAS for the number of matchings in (k,r)-graphs without 3-combs, for any fixed r. In view of this conclusion and Thm. 2, we raise the question if for all $k \leq 5$ and r there is an FPTAS (or at least FPRAS) for the problem $\#M(k,r)$. The first open instance is that of $(3,4)$-graphs. For $k = 4, 5$, to avoid recurrences of depth $k-1 \geq 3$, as an intermediate step, one could first consider the restriction of the class of (k,r)-graphs to those without a 4-comb, that is, to those whose intersection graphs are 4-claw-free. Here, the first open instance is that of $(4,3)$-graphs without 4-combs. In general, it would be also very interesting to elucidate the status of the problem for arbitrary k-graphs for $k = 3, 4$ and 5, or for some generic subclasses of them.

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Appendix: Mathematica expressions

First we define $F(\mathbf{z})$ function:

```
F[z1_, z2_, z3_, z4_, z14_, z13_, z23_, z24_]:=\
1/4(1+z1^4+z2^4+z3^4+z4^4+\
      1/2(z14^4(z1^4+z4^4)+z13^4(z1^4+z3^4)+\
      z23^4(z2^4+z3^4)+z24^4(z2^4+z4^4)))^(-5/4)\
  (2z1^3(2+z14^4+z13^4)+2z2^3(2+z23^4+z24^4)+\
   2z3^3(2+z13^4+z23^4)+2z4^3(2+z14^4+z24^4)+\
   2z14^3(z1^4+z4^4)+2z13^3(z1^4+z3^4)+\
   2z23^3(z2^4+z3^4)+2z24^3(z2^4+z4^4))
```

Next we find the absolute maximum:

```
NMaximize[{F[z1, z2, z3, z4, z14, z13, z23, z24],\
     0<=z1<=1 && 0<=z2<=1 && 0<=z3<=1 && 0<=z4<=1 &&
     0<=z14<=1 && 0<=z13<=1 && 0<=z23<=1 && 0<=z24<=1},\
     {z1, z2, z3, z4, z14, z13, z23, z24}]
```

obtaining that

$$F(\mathbf{z}) \leq F(\zeta, \zeta, \zeta, \zeta, 1, 1, 1, 1) \sim 0.970247,$$

where $\zeta \sim 0.695347$.