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Turán and Ramsey numbers for 3-uniform minimal paths of length 4

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Abstract
We determine Turán numbers for the family of 3-uniform minimal paths of length four *for all* n. We also establish the second- and third-order Turán numbers and use them to compute the corresponding Ramsey numbers for up to four colors.

KEYWORDS
hypergraphs, paths, Ramsey number, Turán number

1 | INTRODUCTION

Turán-type problems concern the maximum number of edges in a (hyper)graph without certain forbidden substructures. They are central to extremal combinatorics and have a long and influential history initiated by Turán in 1944 [24] who solved the problem for all complete graphs. A few years later Erdős and Stone [5] determined asymptotically the Turán numbers for all nonbipartite graphs. Such questions for hypergraphs are, however, notoriously difficult in general, and several natural problems are wide open, most notably Turán's conjecture for the tetrahedron. And, again, asymptotic results are perhaps a little easier to obtain. For a comprehensive survey on Turán numbers for hypergraphs see [16].

Similar stature and even longer history are enjoyed by Ramsey Theory, started by Ramsey's paper [23] and developed in the mid-30s of the 20th century by Erdős and Szekeres [4]. Here the object of interest is the smallest order of a complete (hyper)graph which, when edge-partitioned into a given number of colors, possesses a desired substructure entirely in one color. When the substructure is itself complete, an exact solution of this problem is still beyond our reach already for graphs and becomes hopeless for hypergraphs, except for some very small cases.

The two problems are immanently related by a (trivial) observation that if the number of edges of one color exceeds the Turán number for the target substructure, then there is a monochromatic copy of it in that color. However, since one is typically interested in a small number of colors (as we are), the corresponding Turán numbers should also be known for small number of vertices.

In general, both, Turán and Ramsey problems are more difficult for dense hypergraphs. Consequently, the area of research interest has broadened to include sparser structures like paths and cycles. In this paper we focus on a particular family of 3-uniform hypergraphs, minimal paths of length four, for which the Turán numbers have been already determined *for large* n in [7]. We compute them *for all* n and, consequently, obtain the corresponding Ramsey numbers for up to four colors.

1.1 | Basic definitions

For $k \geq 2$, a k-graph (k-uniform hypergraph) is an ordered pair $H = (V, E)$, where $V = V(H)$ is a finite set (of vertices) and $E = E(H)$ is a subset of the set $\binom{V}{k}$ of k-element subsets of V (called edges). If $E = \binom{V}{k}$, we call H *complete* and denote by $K_n^{(k)}$, where $n = |V(H)|$.

For k-graphs H' and H we say that H' is a sub-k-graph of H and write $H' \subseteq H$ if $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$. Given a family of k-graphs $\mathcal{F}$, we call a k-graph H $\mathcal{F}$*-free* if for all $F \in \mathcal{F}$ we have $F \nsubseteq H$, that is, no sub-k-graph of H is isomorphic to F. Given a family of k-graphs $\mathcal{F}$ and an integer $n \geq 1$, the *Turán number for* $\mathcal{F}$ *and* n is defined as

$$\mathrm{ex}_k(n; \mathcal{F}) := \max\{|E(H)|: |V(H)| = n \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$

Every n-vertex $\mathcal{F}$-free k-graph with exactly $\mathrm{ex}_k(n; \mathcal{F})$ edges is called *extremal for* $\mathcal{F}$. We denote by $\mathrm{Ex}_k(n; \mathcal{F})$ the family of all n-vertex k-graphs which are extremal for $\mathcal{F}$. In the case when $\mathcal{F} = \{F\}$, we will often write $\mathrm{ex}_k(n; F)$ for $\mathrm{ex}_k(n; \{F\})$ and $\mathrm{Ex}_k(n; F)$ for $\mathrm{Ex}_k(n; \{F\})$.

Let $\mathcal{F}$ be a family of k-graphs and $r \geq 2$ be an integer. The *Ramsey number* $R(\mathcal{F}; r)$ is the smallest integer n such that every r-edge-coloring of the complete k-graph $K_n^{(k)}$ yields a monochromatic copy of a member of $\mathcal{F}$. The relationship between Turán and Ramsey numbers allured to above is best exemplified by the following implication:

$$\frac{1}{r}\binom{n}{k} > \mathrm{ex}_k(n; \mathcal{F}) \Rightarrow R(\mathcal{F}; r) \leq n. \tag{1}$$

As mentioned earlier, we shall consider the Turán problem for a special family of 3-uniform paths. At this point the reader should be alerted that there are several other notions of paths and cycles in k-graphs (e.g., Berge, loose, linear, and tight) and that authors take a great liberty in using those names (except for Berge). In this paper we restrict our attention to minimal paths and cycles defined as follows.

Given $k, \ell \geq 2$, a k-uniform *minimal* ℓ*-path* (a.k.a. *loose*) is a k-graph with edge set $\{a_0, a_1, ..., a_{\ell-1}\}$ such that $a_i \cap a_j \neq \emptyset$ if and only if $|i - j| \leq 1$, while a k*-uniform minimal* ℓ*-cycle* is a k-graph with edge set $\{a_0, a_1, ..., a_{\ell-1}\}$ such that $a_i \cap a_j \neq \emptyset$ if and only if $|i - j| \leq 1 \pmod{\ell}$. So, minimal paths and cycles form special subclasses of, respectively, Berge paths and cycles (see, e.g., [19]), with no redundant edge intersections. Put another way, the minimality manifests itself by no vertex belonging to more than two edges.

We write $\mathcal{P}_\ell^{(k)}$ for the family of all k-uniform minimal ℓ-paths and $\mathcal{C}_\ell^{(k)}$ for the family of all k-uniform minimal ℓ-cycles (see Figure 1 for all 3-uniform minimal 4-paths). Note that the

FIGURE 1 All 3-uniform minimal 4-paths from $\mathcal{P}4^{(3)}$ [Color figure can be viewed at wileyonlinelibrary.com]

longest path in $\mathcal{P}_\ell^{(k)}$ has $\ell(k-1)+1$ vertices. It is called *linear* (a.k.a. *loose*), since edges intersect pairwise in at most one vertex, and denoted by $P_\ell^{(k)}$. For convenience, in what follows we shall write $\mathcal{P}_4$ instead of $\mathcal{P}4^{(3)}$. For $k=2$ the families $\mathcal{P}_\ell^{(2)}$ and $\mathcal{C}_\ell^{(2)}$ each consists of a single graph, the ordinary (graph) path and cycle, which will be denoted by, respectively, $P_\ell^{(2)}$ and $C_\ell^{(2)}$.

1.2 | Main results

Mubayi and Verstraëte [19] showed that $\mathrm{ex}_k(n; \mathcal{P}_3^{(k)}) = \binom{n-1}{k-1}$ for all $n \geq 2k$ and $\mathrm{ex}_3(n; \mathcal{P}_\ell^{(3)}) \leq \frac{5\ell-1}{6}\binom{n-1}{2}$ for all $n \geq 3(\ell+1)/2$. Füredi, Jiang, and Seiver [7] proved that, for $k \geq 3$, $t \geq 1$, and *for sufficiently large* n,

$$\mathrm{ex}_k\left(n; \mathcal{P}_{2t+1}^{(k)}\right) = \binom{n}{k} - \binom{n-t}{k} \quad \text{and} \quad \mathrm{ex}_k\left(n; \mathcal{P}_{2t+2}^{(k)}\right) = \binom{n}{k} - \binom{n-t}{k} + 1, \tag{2}$$

and that the unique extremal k-graph consists of all k-tuples intersecting a given set T of t vertices plus, for even ℓ, one extra edge disjoint from T. Note that for $t=1$, the above expressions become, respectively, $\binom{n-1}{k-1}$ and $\binom{n-1}{k-1}+1$.

In fact, in [7] the authors focused on linear paths and determined Turán numbers $\mathrm{ex}_k(n; P_\ell^{(k)})$ for large n and $k \geq 4$, while Kostochka, Mubayi, and Verstraëte [17] did the same for large n and $\ell \geq 4$. The remaining case of $\ell = k = 3$ was also implicit in their proof, but again for large n. In [14], it was proved for all $n \geq 7$ that $\mathrm{ex}_3\left(n; P_3^{(3)}\right) = \binom{n-1}{2}$. As for the other 3-minimal path, called *messy* by Bohman and Zhu in [2] and defined as $M_3 = \{abc, bcd, def\}$, it was proved therein that $\mathrm{ex}_3(n; M_3) = \binom{n-1}{2}$ for all $n \geq 6$.

In this paper we similarly extend (2) in the smallest open case, that is, we determine the Turán numbers $\mathrm{ex}_3(n; \mathcal{P}_4)$ for *all* n. All special 3-graphs appearing in Theorem 1.1, as well as in Theorems 1.3–1.5 in Section 1.3, are defined, for clarity of exposition, only in Section 2.

Theorem 1.1. *For* $n \geq 1$,

$$\mathrm{ex}_3(n; \mathcal{P}_4) = \begin{cases} \binom{n}{3} & \text{and } \mathrm{Ex}_3(n; \mathcal{P}_4) = \{K_n\} & \text{for } n \leq 6, \\ 20 & \text{and } \mathrm{Ex}_3(n; \mathcal{P}_4) = \left\{K_6^{(3)} \cup K_1\right\} & \text{for } n = 7, \\ 22 & \text{and } \mathrm{Ex}_3(n; \mathcal{P}_4) = \left\{S_8^{+1}, SP_8, SK_8\right\} & \text{for } n = 8, \\ \binom{n-1}{2} + 1 & \text{and } \mathrm{Ex}_3(n; \mathcal{P}_4) = \left\{S_n^{+1}\right\} & \text{for } n \geq 9. \end{cases}$$

(Note that for $n = 8$, we have $\binom{n-1}{2} + 1 = 22$).

At this point it is worth looking at the current 'Turán status' of the four individual members of the minimal family $\mathcal{P}_4$, pictured in Figure 1. For the one on top left, the linear 4-path P_4, it was shown in [17] that, for large n, $\mathrm{ex}_3(n; P_4) = \binom{n-1}{2} + n - 3$. For the one on top right, called the $(2, 1)$-path and denoted by $P(2, 1)$ by Füredi, Jiang, Kostochka, Mubayi, and Verstraëte in [6], it was shown only that $\mathrm{ex}_3(n; P(2, 1)) = \binom{n-1}{2} + o(n^2)$. Seemingly symmetrical 3-graph on the bottom left, called the $(1, 2)$-path and denoted by $P(1, 2)$, turned out to be harder. It was predicted in [6], as a special case of a more general conjecture, that the same asymptotic formula as for $P(2, 1)$ holds also for $P(1, 2)$. Very recently, this prediction was confirmed by Füredi and Kostochka in [8]. In fact, they showed that $\mathrm{ex}_3(n; P(1, 2)) = \binom{n-1}{2} + O(n)$. The last of the minimal 4-paths, the one on the bottom right in Figure 1, let us call M_4, extends the messy 3-path mentioned above by one edge. So far we have no tools to approach the problem of finding the Turán number for M_4.

As an immediate consequence of Theorem 1.1 and the relation (1), plugging $n = 3r + 1$, we infer that, for $r \geq 3$, $R(\mathcal{P}_4; r) \leq 3r + 1$. On the other hand, a simple construction originated in [10] (see Section 7 for more details) yields a lower bound $R(\mathcal{P}_4; r) \geq r + 6$ for all $r \geq 1$. Using Theorem 1.1 along with some more technical results from Section 1.3, we confirm that, at least for up to four colors, the lower bound is, indeed, the correct value.

Theorem 1.2. *For $r \leq 4$, we have $R(\mathcal{P}_4; r) = r + 6$.*

1.3 | Turán numbers of higher orders

To calculate Ramsey numbers based on Turán numbers, it is sometimes necessary to consider Turán numbers of higher orders (see, e.g., [15]), which can be defined iteratively as follows. The Turán number of the first order is the ordinary Turán number. For a family of k-graphs $\mathcal{F}$ and integers $s, n \geq 1$, *the Turán number of the* $(s + 1)$*st order* is defined as

$$\mathrm{ex}_k^{(s+1)}(n; \mathcal{F}) = \max\Big\{|E(H)|\colon |V(H)| = n,\ H \text{ is } \mathcal{F}\text{-free, and}$$
$$\forall\, H' \in \mathrm{Ex}_k^{(1)}(n; \mathcal{F}) \cup \cdots \cup \mathrm{Ex}_k^{(s)}(n; \mathcal{F}), H \nsubseteq H'\Big\},$$

if such a k-graph H exists. An n-vertex $\mathcal{F}$-free k-graph H is called $(s+1)$*-extremal for*$\mathcal{F}$ if $|E(H)| = \mathrm{ex}_k^{(s+1)}(n; \mathcal{F})$ and $\forall\, H' \in \mathrm{Ex}_k^{(1)}(n; \mathcal{F}) \cup \cdots \cup \mathrm{Ex}_k^{(s)}(n; \mathcal{F}), H \nsubseteq H'$; we denote by $\mathrm{Ex}_k^{(s+1)}(n; \mathcal{F})$ the family of n-vertex k-graphs which are $(s + 1)$-extremal for $\mathcal{F}$.

A historically first example of a Turán number of the second order is due to Hilton and Milner [12] who determined the maximum size of *a nontrivial* intersecting k-graph, that is, one which is not a star (see the definition in Section 2). Recall that a 3-graph is intersecting if and only if it is M_2-free and that, by Erdős–Ko–Rado theorem [3], $\mathrm{ex}(n, M_2) = \binom{n-1}{2}$ for $n \geq 6$, while for $n \geq 7$ the only extremal 3-graph is a full star. Hilton and Milner proved that

Theorem 1.3 (Hilton and Milner [12]). *For $n \geq 7$ we have $\mathrm{ex}_3^{(2)}(n; M_2) = 3n - 8$.*

In [11] the authors determined $\mathrm{ex}_k^{(3)}(n; M_2^{(k)})$ for all k; in [22] the complete hierarchy of 3-uniform Turán numbers $\mathrm{ex}_3^{(s)}(n; M_2)$, $s = 1, \ldots, 6$, has been found (for $s \geq 7$ they do not exist).

In this paper we determine for $\mathcal{P}_4$ the Turán numbers of the second- and third-order.

Theorem 1.4. *For* $n \geq 9$,

$$\mathrm{ex}_3^{(2)}(n; \mathcal{P}_4) = \begin{cases} 5n - 18 & \text{and} \quad \mathrm{Ex}_3^{(2)}(n; \mathcal{P}_4) = \{SP_n\} \quad \text{for} \quad n \leq 11, \\ \binom{n-3}{2} + 7 & \text{and} \quad \mathrm{Ex}_3^{(2)}(n; \mathcal{P}_4) = \{CB_n\} \quad \text{for} \quad n \geq 12. \end{cases}$$

Theorem 1.5. *For* $n \geq 9$,

$$\mathrm{ex}_3^{(3)}(n; \mathcal{P}_4) = \begin{cases} 4n - 10 & \text{and} \quad \mathrm{Ex}_3^{(3)}(n; \mathcal{P}_4) = \{SK_n\} \quad \text{for} \quad n \leq 10, \\ \binom{n-3}{2} + 7 = 35 & \text{and} \quad \mathrm{Ex}_3^{(3)}(n; \mathcal{P}_4) = \{CB_n\} \quad \text{for} \quad n = 11, \\ 5n - 18 = 42 & \text{and} \quad \mathrm{Ex}_3^{(3)}(n; \mathcal{P}_4) = \{SP_n\} \quad \text{for} \quad n = 12, \\ 47 & \text{and} \quad \mathrm{Ex}_3^{(3)}(n; \mathcal{P}_4) = \{SP_n, B_n\} \quad \text{for} \quad n = 13, \\ \binom{n-4}{2} + 11 & \text{and} \quad \mathrm{Ex}_3^{(3)}(n; \mathcal{P}_4) = \{B_n\} \quad \text{for} \quad n \geq 14. \end{cases}$$

Note that for $n = 13$ we have $5n - 18 = \binom{n-4}{2} + 11 = 47$.

1.4 | Notation

For a k-graph H and a vertex $v \in V(H)$, the *link graph* of v in H is the $(k-1)$-graph on the vertex set $V(H)$ and the edge set

$$L_H(v) = \{e \setminus \{v\} : v \in e \in H\}.$$

The degree of v in H is defined as $\deg_H(v) = |L_H(v)|$, while maximum and minimum degrees in H are denoted by $\Delta_1(H)$ and $\delta_1(H)$, respectively. For $k = 2$, we obtain the ordinary notions of degrees and maximum and minimum degrees in a graph. Also, in the case $k = 2$, the link graph is just a set of singletons and coincides with the standard notion of the neighborhood $N_G(v)$. The subscript $_1$ in $\Delta_1(H)$ and $\delta_1(H)$ is often omitted.

For a 3-graph H on V, the set of neighbors of a pair $x, y \in V$ in H is defined as

$$N_H(x, y) = \{z : \{x, y, z\} \in H\}.$$

The number $\deg_H(x, y) = |N_H(x, y)|$ is called *degree of the pair of vertices* x, y and we set $\Delta_2(H) = \max_{x,y \in V} \deg_H(x, y)$ for the *maximum pair degree* in H.

We identify a k-graph H with its edge set $E(H)$. Throughout the paper we will use the name "edge" for both, the edges of a 3-graph (triples) and the edges of a 2-graph (pairs). It will always be clear from the context which one is meant. For a k-graph H with vertex set V we write

$$V[H] := \bigcup_{h \in H} h$$

for the set of all nonisolated vertices, that is, vertices v with $\deg_H(v) > 0$. Given $W \subseteq V$ we write

$$H[W] := \{h \in H : h \subseteq W\}$$

for the sub-k-graph of H induced by W.

For simplicity, if there is no danger of confusion, we sometimes denote edges $\{x, y\}$ of graphs and edges $\{x, y, z\}$ of 3-graphs by xy and xyz, respectively. Also, if $f = \{x, y\}$ is a pair of vertices and $v \in V$ is a single vertex, we may write fv for the edge $\{x, y, v\} \in H$.

Notation $f_1 f_2 \cdots f_\ell$ will represent a minimal path with edges $f_1, f_2, \ldots, f_\ell$ in this order and, likewise, notation $v_1 v_2 \cdots v_m$ will represent a minimal path with vertices $v_1, v_2, \ldots, v_m$ in this order. The same shorthand notation may apply to cycles as well.

For two k-graphs G, H, let $G \cup H$ denote the disjoint union of them. If H is a k-graph on V, $v \in V$, and $e \in H$ is an edge of H, then we denote by $H - v$ the k-graph obtained from H by deleting vertex v together with all edges containing it, whereas by $H - e$ we mean the k-graph obtained from H by deleting the single edge e. For a k-graph H, by H^c we mean the complement of H, that is, $H^c = \binom{V}{k} \setminus H$.

1.5 | Organization

The rest of the paper is organized as follows. In Section 2 we construct 3-graphs which play a special role in the statements and proofs of our results. In Section 3 we introduce several lemmas and use them to deduce Theorems 1.1, 1.4, and 1.5. The proofs of these lemmas are presented in Sections 4–6. We prove Theorem 1.2 in Section 7. This proof relies only on the statements of Theorems 1.1, 1.4, and 1.5, and thus can be understood without reading the earlier sections. Finally, Section 8 contains a couple of open problems.

2 | SPECIAL 3-GRAPHS

In this section we define 3-graphs which play a special role in the paper, either as tools in the proofs or as extremal 3-graphs. By default, we drop the superscript $^{(3)}$.

The (unique) 6-vertex minimal 4-cycle C_4 is a 3-graph with

$$V(C_4) = \{x_1, x_2, y_1, y_2, z_1, z_2\} \quad \text{and} \quad E(C_4) = \{x_1 y_1 y_2, y_1 y_2 x_2, x_2 z_1 z_2, z_1 z_2 x_1\}$$

(see Figure 2A). Further, let $K := K_4$ stand for the complete 3-graph on four vertices and let $P := P_2$ denote the minimal 2-path with five vertices, that is, two edges sharing one vertex.

For $s \geq 2$, let M_s stand for the matching of size s, that is, a 3-graph consisting of s disjoint edges.

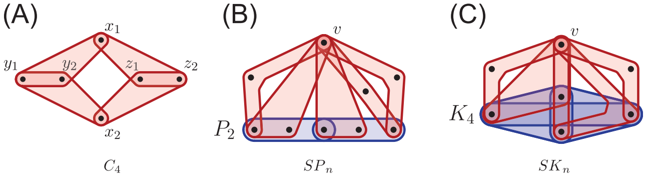


FIGURE 2 Four-cycle C_4, SP_n, and SK_n: (A) C_4, (B) SP_n, and (C) SK_n [Color figure can be viewed at wileyonlinelibrary.com]

2.1 | Stars

A *star* is a 3-graph S with a vertex v (called sometimes the center) contained in all the edges of S. A star is *full* if it consists of all sets in $\binom{V}{3}$ containing v, that is, if $\deg_S(v) = \binom{|V|-1}{2}$. Normally, we write S_n for the full star with n vertices, but if we want to specify the vertex set and the star center, we may sporadically use symbol S_V^v instead. By S_n^{+1} we denote the unique (up to isomorphism) n-vertex 3-graph obtained from the full star S_n by adding one extra edge. We call S_n^{+1} a *starplus*.

2.2 | *F*-stars

For a set V of $n \geq 6$ vertices, a subset $A \subset V$, and a vertex $v \in V \setminus A$, let $S(v, A) = S_V^v \setminus S_{V\setminus A}^v$ be the star obtained from the full star S_V^v by deleting all edges disjoint from A. In other words, $S(v, A)$ consists of all triples containing v and at least one vertex of A.

Given a 3-graph F, we define the F-star by $SF_n := F \cup S(v, V(F))$, where $V \supset V(F)$, $|V| = n$, and $v \in V \setminus V(F)$. We will focus on two instances of F-stars: with $F = P$ and $F = K$ (see Figure 2B,C). It is easy to check that both, SK_n and SP_n, are $\{\mathcal{P}_4, M_3\}$-free and contain a copy of C_4. Moreover, $|SK_n| = 4n - 10$ and $|SP_n| = 5n - 18$. Notice that for $n = 8$ these two expressions are equal to each other.

2.3 | Balloons

Finally, we define two more deformations of stars. For $n \geq 9$, let B_n be a 3-graph on n vertices, called the *balloon*, obtained from the full star S_{n-3} with center x by selecting three vertices $y_1, y_2, y_3 \in V(S_{n-3}) \setminus \{x\}$, adding three new vertices z_1, z_2, z_3, and adding eleven new edges: $\{y_1, y_2, y_3\}$, $\{z_1, z_2, z_3\}$, and all nine edges of the form $\{x, y_i, z_j\}$, $i, j = 1, 2, 3$ (see Figure 3A). Note that the balloon B_n is $\mathcal{P}_4$-free, contains M_3, and has $\binom{n-4}{2} + 11$ edges.

For $n \geq 8$, let CB_n be a 3-graph on n vertices, called the *compact balloon*, obtained from the full star S_{n-2} with center x by selecting two vertices $y_1, y_2 \in V(S_{n-2}) \setminus \{x\}$, adding two new vertices z_1, z_2, and adding seven new edges: $\{y_1, y_2, z_1\}$, $\{y_1, y_2, z_2\}$, all four edges of the form

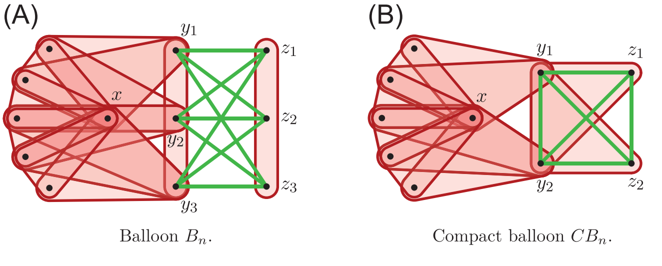


FIGURE 3 Balloons. The green pairs form 3-edges with the vertex x: (A) balloon $B_n$and (B) compact balloon CB_n [Color figure can be viewed at wileyonlinelibrary.com]

$\{x, y_i, z_j\}$, $i, j = 1, 2$, and the edge $\{x, z_1, z_2\}$ (see Figure 3B). Note that the compact balloon CB_n is $\mathcal{P}_4$-free, is not a sub-3-graph of the starplus S_n^{+1}, and has $\binom{n-3}{2} + 7$ edges.

3 | TURÁN NUMBERS

The goal of this section is to prove Theorems 1.1, 1.4, and 1.5. To do this we divide the family of all $\mathcal{P}_4$-free 3-graphs into some special subfamilies and then count the maximum number of edges within them separately (see Figure 4).

Next, we compare to each other bounds obtained in Lemmas 3.2–3.6. For $n \geq 14$ we have

$$4n - 10 < 5n - 18 < \binom{n-4}{2} + 11 < \binom{n-3}{2} + 7 < \binom{n-1}{2} + 1, \tag{3}$$

whereas for $n \in [8, 14]$ we gather these bounds in Table 1.

However, before we do this precisely, we need one more piece of notation. A 3-graph H is said to be *connected* if for every partition of the vertex set $V(H) = U \cup W$, there is an edge in H with nonempty intersection with both subsets, U and V.

A forced presence of a sub-k-graph can be expressed in terms of conditional Turán numbers, introduced in [14]. For a k-graph F, an F-free k-graph G, and an integer $n \geq |G|$, the *conditional Turán number* is defined as

$$\mathrm{ex}_k(n; F|G) = \max\{|E(H)| : |V(H)| = n,\ H \text{ is } F\text{-free, and } H \supseteq G\}.$$

Every n-vertex F-free k-graph H with $\mathrm{ex}_k(n; F|G)$ edges and such that $H \supseteq G$ is called *G-extremal for F*. We denote by $\mathrm{Ex}_k(n; F|G)$ the family of all n-vertex k-graphs which are

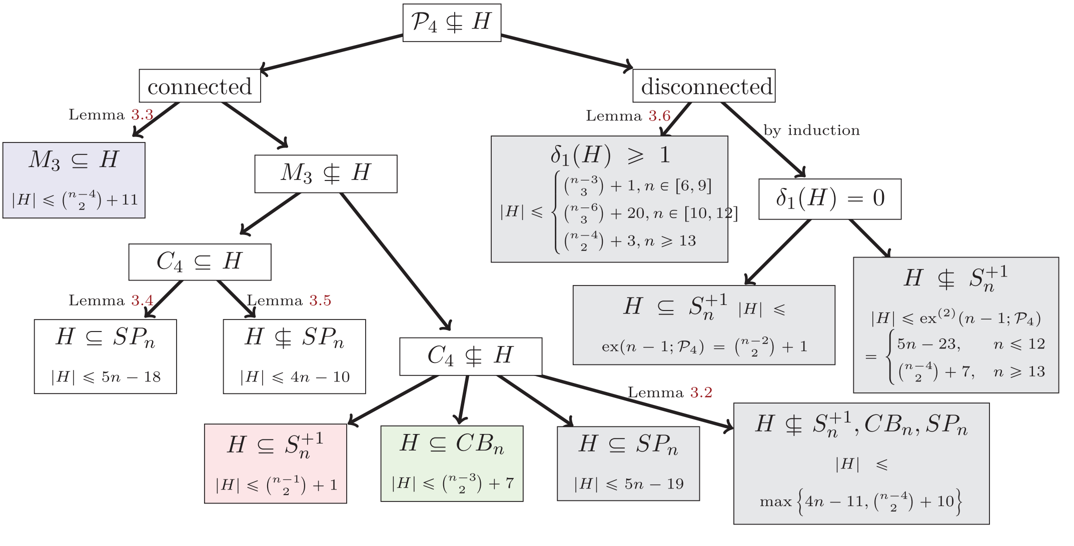


FIGURE 4 Division of the family of $\mathcal{P}_4$-free 3-graphs. The gray blocks contain 3-graphs not appearing in extremal families of the first three orders. For $n \geq 14$ the red, green, and blue block represent, respectively, the first-, second-, and third-order Turán number for $\mathcal{P}_4$ [Color figure can be viewed at wileyonlinelibrary.com]

TABLE 1 The Turán numbers for $\mathcal{P}_4$ and $n \in [9, 14]$ of the first, second- and third-order

n	S_n^{+1} $\binom{n-1}{2}+1$	SP_n $5n-18$	SK_n $4n-10$	CB_n $\binom{n-3}{2}+7$	B_n $\binom{n-4}{2}+11$	$\max\{4n-11, \binom{n-4}{2}+10\}$	Disconn. Lemma 3.6
8	22	22	22	17	17	21	11
9	29	27	26	22	21	25	21
10	37	32	30	28	26	29	24
11	46	37	34	35	32	33	30
12	56	42	38	43	39	38	40
13	67	47	42	52	47	46	39
14	79	52	46	62	56	55	48

G-extremal for F. For $k = 3$ we drop the subscript$_3$. The *conditional Turán number of the* sth order is defined in a similar way as the ordinary Turán number of the sth order (see Section 1.3). Finally, for $k = 3$, if in the above definition one restricts oneself to connected 3-graphs, we add the subscript $_{conn}$ and denote the corresponding extremal numbers and families, respectively, by $\mathrm{ex}_{conn}(n; F|G)$, $\mathrm{Ex}_{conn}(n; F|G)$, $\mathrm{ex}^{(s)}_{conn}(n; F|G)$, and $\mathrm{Ex}^{(s)}_{conn}(n; F|G)$.

Now let us state a few lemmas from which Theorems 1.1, 1.4, and 1.5 follow. The case $n = 7$ is treated separately.

Lemma 3.1. $\mathrm{ex}(7; \mathcal{P}_4) = 20$, $\mathrm{Ex}(7; \mathcal{P}_4) = \left\{K_6^{(3)} \cup K_1\right\}$.

Lemma 3.2. *Let H be a $\{\mathcal{P}_4, C_4, M_3\}$-free connected 3-graph on $n \geq 8$ vertices. If $H \nsubseteq S_n^{+1}$, $H \nsubseteq SP_n$, and $H \nsubseteq CB_n$ then*

$$|H| \leq \max\left\{4n - 11, \binom{n-4}{2} + 10\right\}.$$

Lemma 3.3. *For $n \geq 9$,* $\mathrm{ex}_{conn}(n; \mathcal{P}_4|M_3) = \binom{n-4}{2} + 11$ *and the balloon B_n is the only extremal 3-graph.*

Lemma 3.4. *For $n \geq 8$,*

$$\mathrm{ex}_{conn}(n; \mathcal{P}_4 \cup \{M_3\}|C_4) = 5n - 18,$$
$$\mathrm{Ex}_{conn}(n; \mathcal{P}_4 \cup \{M_3\}|C_4) = \begin{cases} \{SP_8, SK_8\} & \text{for} \quad n = 8, \\ \{SP_n\} & \text{for} \quad n \geq 9. \end{cases}$$

Lemma 3.5. *For $n \geq 9$,*

$$\mathrm{ex}^{(2)}_{conn}(n; \mathcal{P}_4 \cup \{M_3\}|C_4) = 4n - 10,$$
$$\mathrm{Ex}^{(2)}_{conn}(n; \mathcal{P}_4 \cup \{M_3\}|C_4) = \{SK_n\}.$$

Lemma 3.6. *If H is a disconnected $\mathcal{P}_4$-free 3-graph on n vertices, with $\delta_1(H) \geq 1$, then*

$$|H| \le \begin{cases} \binom{n-3}{3} + 1 & \text{for} \quad 6 \le n \le 9, \\ \binom{n-6}{3} + 20 & \text{for} \quad 10 \le n \le 12, \\ \binom{n-4}{2} + 3 & \text{for} \quad n \ge 13. \end{cases}$$

Proof. Let H_1 be a connected component of H with the smallest number of vertices. Set $H_2 = H \backslash H_1$, $n_i = |V[H_i]|$, $i = 1, 2$. Clearly $3 \le n_1 \le n_2 = n - n_1 \le n - 3$.

We argue by induction on n. For the base case $6 \le n \le 9$, we use the fact that $|H_i| \le |K_{n_i}^{(3)}| = \binom{n_i}{3}$, $i = 1, 2$. Therefore, a simple optimization shows

$$|H| = |H_1| + |H_2| \le \binom{n_1}{3} + \binom{n_2}{3} \le 1 + \binom{n-3}{3},$$

as required.

For the induction step assume $n \ge 10$ and that Lemma 3.6 is true for all disconnected $\mathcal{P}_4$-free 3-graphs with less than n vertices and $\delta_1(H) \ge 1$. Then, as $n_2 \le n - 3$ we are in a position to apply the induction hypothesis to H_2 in case it is disconnected. For H_1, as well as, for connected H_2 we apply Lemmas (3.1)–(3.4). Altogether, we claim that, for $i = 1, 2$,

$$|H_i| \le \begin{cases} \binom{n_i}{3} & \text{for} \quad n_i \le 6, \\ 19 & \text{for} \quad n_i = 7, \\ \binom{n_i-1}{2} + 1 & \text{for} \quad n_i \ge 8. \end{cases} \tag{4}$$

Indeed, for $n_i \le 6$ clearly $|H_i| \le |K_{n_i}^{(3)}| \le \binom{n_i}{3}$, whereas for $n_i = 7|H_i| \le 19$ follows from Lemma 3.1 combined with $\delta_1(H) \ge 1$. Finally, to show that $|H_i| \le \binom{n_i-1}{2} + 1$ for $n_i \ge 8$, in view of Lemmas 3.2, 3.3, 3.4, and 3.6 (see also Figure 4) it is enough to observe that

$$\binom{n-1}{2} + 1 > \begin{cases} \binom{n-3}{3} + 1 & \text{for} \quad n \le 10, \\ \binom{n-6}{3} + 20 & \text{for} \quad 8 \le n \le 14, \\ \max\left\{\binom{n-3}{2} + 7, \binom{n-4}{2} + 11, 5n - 19\right\} & \text{for} \quad n \ge 8. \end{cases} \tag{5}$$

In particular, for $n \ge 8$, $\binom{n-1}{2} + 1 > 4n - 11$, as well as, $\binom{n-1}{2} + 1 \ge 5n - 18$.

Now we use (4) to bound the number of edges in H. Considering separately cases $n_1 = 3, 4, ..., \lfloor n/2 \rfloor$ one gets,

$$|H| \le \begin{cases} \max\{1 + 19, 4 + 20, 10 + 10\} = \binom{n-6}{3} + 20 & \text{for} \quad n = 10, \\ \max\{1 + 22, 4 + 19, 10 + 20\} = \binom{n-6}{3} + 20 & \text{for} \quad n = 11, \\ \max\{1 + 29, 4 + 22, 10 + 19, 20 + 20\} = \binom{n-6}{3} + 20 & \text{for} \quad n = 12, \\ \max\{1 + 37, 4 + 29, 10 + 22, 20 + 19\} = \binom{n-4}{2} + 3 & \text{for} \quad n = 13. \end{cases}$$

Therefore it remains to take care of $n \geq 14$. If $n_1 \leq 6$, then $n_2 \geq 8$ and thus $|H_1| \leq \binom{n_1}{3}$, $|H_2| \leq \binom{n_2-1}{2} + 1$, yielding

$$\begin{aligned}|H| &\leq \max\left\{\binom{n-4}{2} + 2, \binom{n-5}{2} + 5, \binom{n-6}{2} + 11, \binom{n-7}{2} + 21\right\}\\ &= \binom{n-4}{2} + 2.\end{aligned}$$

For $n_1 = 7$, $|H_1| \leq 19$ and hence

$$|H| \leq \begin{cases} 19 + 19 < \binom{n-4}{2} + 3 & \text{for} \quad n = 14, \\ \binom{n-8}{2} + 20 < \binom{n-4}{2} + 3 & \text{for} \quad n \geq 15. \end{cases}$$

Finally, if $n_1 \geq 8$, then also $n_2 \geq 8$ and thus $|H_i| \leq \binom{n_i}{2} + 1$, $i = 1, 2$. But then, clearly

$$|H| = |H_1| + |H_2| \leq \binom{n_1 - 1}{2} + \binom{n_2 - 1}{2} + 2 < \binom{n-4}{2} + 3.$$

□

Now we are ready to prove Theorems 1.1, 1.4, and 1.5.

Proof of Theorem 1.1. We argue by induction on n. For the base case $n \leq 6$ the assumption easily follows from the fact that every minimal 4-path has at least 7 vertices, whereas for $n = 7$ we use Lemma 3.1.

Next, we let $n \geq 8$ and observe that as S_n^{+1} is a $\mathcal{P}_4$-free 3-graph with $\binom{n-1}{2} + 1$ edges, we get

$$\text{ex}(n; \mathcal{P}_4) \geq \binom{n-1}{2} + 1.$$

To obtain the reverse bound on $\text{ex}(n; \mathcal{P}_4)$ we let H to be a $\mathcal{P}_4$-free 3-graph on $n \geq 8$ vertices and with at least $\binom{n-1}{2} + 1$ edges. We argue that $H = S_n^{+1}$ for $n \geq 9$, whereas for $n = 8, H = S_n^{+1}$ or $H \in \{SP_8, SK_8\}$, which will end the proof. To this end we consider separately connected and disconnected $\mathcal{P}_4$-free 3-graphs. In the former case Lemma 3.3 together with $\binom{n-4}{2} + 11 < \binom{n-1}{2} + 1$ tells us that $M_3 \nsubseteq H$. Further, as for $n \geq 8$ we have $5n - 18 \leq \binom{n-1}{2} + 1$ with the equality only for $n = 8$, in view of Lemma 3.4 we learn that for $n \geq 9, H$ is C_4-free, whereas for $n = 8$ the only possibility to have $C_4 \subseteq H$ is $H \in \{SP_8, SK_8\}$. Finally we use Lemma 3.2 to deduce that the only $\{\mathcal{P}_4, C_4, M_3\}$-free 3-graph with at least $\binom{n-1}{2} + 1$ edges is S_n^{+1}, as required (see Figure 4, Table 1, and Equation 3).

Now, to exclude the disconnected case we first assume that $\delta_1(H) \geq 1$ and use Lemma 3.6 combined with (5). Finally, if H contains an isolated vertex v, then we can apply the induction hypothesis to $H - v$, obtaining

$$|H| = |H - v| \leq \text{ex}(n - 1; \mathcal{P}_4) < \binom{n-1}{2} + 1,$$

which ends the proof. □

Proof of Theorem 1.4. The proof is similar to the proof of Theorem 1.1. Let H be a $\mathcal{P}_4$-free 3-graph on the set of vertices V, $|V| = n \geq 9$ with $|H| = \mathrm{ex}^{(2)}(n; \mathcal{P}_4)$. Moreover, as we are computing the second-order Turán number and $\mathrm{Ex}(n; \mathcal{P}_4) = \{S_n^{+1}\}$ for $n \geq 9$, we may assume that $H \nsubseteq S_n^{+1}$. Because both 3-graphs SP_n and CB_n are $\mathcal{P}_4$-free and are not contained in S_n^{+1}, we have the lower bound

$$|H| = \mathrm{ex}^{(2)}(n; \mathcal{P}_4) \geq \max\left\{5n - 18, \binom{n-3}{2} + 7\right\} = \begin{cases} 5n - 18 & \text{for} \quad n \leq 11, \\ \binom{n-3}{2} + 7 & \text{for} \quad n \geq 12. \end{cases} \tag{6}$$

We argue that $H = SP_n$ for $n \leq 11$ and $H = CB_n$ for $n \geq 12$. The proof is by induction on n.

First assume that H is connected and notice that since $\binom{n-4}{2} + 11 < \binom{n-3}{2} + 7$ for $n \geq 9$, Lemma 3.3 yields $M_3 \nsubseteq H$. Therefore, since $4n - 11 < 5n - 18$, in view of Lemmas 3.2 and 3.4 combined with $H \nsubseteq S_n^{+1}$, either $H = SP_n$ or $H = CB_n$, as required (see Figure 4, Table 1, and Equation 3).

In the disconnected case Lemma 3.6 tells us that $\delta_1(H) = 0$, because clearly $\binom{n-4}{2} + 3 < \binom{n-3}{2} + 7$ and for $n \leq 12$ the bound obtained in this lemma is smaller than $5n - 18$ (see Table 1). Thus we let v be an isolated vertex of H. For the base case, $n = 9$ we use Theorem 1.1, getting

$$|H| = |H - v| \leq \mathrm{ex}(8, \mathcal{P}_4) = 22 < 27 = 5n - 18.$$

For the induction step assume $n \geq 10$ and that Theorem 1.4 is true for $n - 1$ in place of n. Now observe, that because $H \nsubseteq S_n^{+1}$, we also have $H - v \nsubseteq S_{n-1}^{+1}$, and consequently,

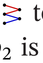
$$|H| = |H - v| \leq \mathrm{ex}^{(2)}(n - 1, \mathcal{P}_4) = \begin{cases} 5n - 23 & \text{for} \quad n \leq 12, \\ \binom{n-4}{2} + 7 & \text{for} \quad n \geq 13, \end{cases}$$

contradicting (6). □

The proof of Theorem 1.5 is very similar to the one of Theorem 1.4, and therefore we left it to the Reader (see Figure 4, Table 1, and Equation 3).

4 | SEVEN VERTICES—PROOF OF LEMMA 3.1

4.1 | Two-colored graphs without a forbidden pattern

In the whole subsection we consider only ordinary 2-graphs, therefore for simplicity of notation we omit the superscript$^{(2)}$ here. We prove two lemmas needed in the proof of Lemma 3.1, where link graphs, R and B, of two given vertices are considered. However, before we state them, one more piece of notation is needed. Let two graphs, R and B, on the same vertex set be given. We define an *rr-bb-path PRB*4 =  to be a subgraph of $R \cup B$ consisting of 4 edges, $r_1, r_2 \in R$ and $b_1, b_2 \in B$, such that $r_1 r_2 b_1 b_2$ is the 4-edge path P_4. By $T \cup \{e\}$ we denote a graph on five vertices consisting of a complete graph on three vertices $T = K_3$ and a single edge e, disjoint from $V[T]$. We start with two technical facts used in further proofs.

Fact 4.1. Let R and B be two graphs on the same 5-vertex set, such that $PRB4 \nsubseteq R \cup B$. If $K_{2,3} \subseteq R$, then $|B| \leq 4$ and either $B \subseteq T \cup \{e\}$ or $|R| + |B| \leq 11$.

Proof. We let $K_{2,3} \subseteq R$ and $B \nsubseteq T \cup \{e\}$, since otherwise $|B| \leq 4$, and the assertion follows. Note that due to $PRB4 \nsubseteq R \cup B$, whenever $|K_{2,3} \cap B| = 1$, then four pairs of $\binom{V}{2}$, shown in Figure 5A with dashed lines, are forbidden for B. In particular $|K_{2,3} \cap B| \leq 1$ causes $B \subseteq T \cup \{e\}$, and thus we may assume $|K_{2,3} \cap B| \geq 2$. Further, $M_2 \subseteq K_{2,3} \cap B$ entails $|B| \leq 3$ (see Figure 5B) and $|B| = 3$ yields $|R| \leq 8$ (see Figure 5C). Therefore in this case, either $B \subseteq T \cup \{e\}$ or $|R| + |B| \leq 11$, as required. Finally, if $P_2 \subseteq K_{2,3} \cap B$, then $B \subseteq T \cup \{e\}$ (see Figure 5D), and the assertion follows again. □

Fact 4.2. Let R and B be two graphs on the same 5-vertex set, such that $PRB4 \nsubseteq R \cup B$. If $C_5 \subseteq R$ and $|R| \geq 6$, then $|B| \leq 4$.

Proof. We let $C_5 \subseteq R$. Now, if $|C_5 \cap B| = 1$ then $|C_5^c \cap B| \leq 3$ and thus $|B| \leq 4$, as required (see Figure 5E). Further, for $|C_5 \cap B| \geq 2$ we have $|B| \leq 3$ (see Figure 5F) and we are done again. Finally, let $C_5 \cap B = \varnothing$, that is, $B \subseteq C_5^c$. Then $|B| \geq 5$ entails $B = C_5^c$ (see Figure 5G). This, in turn, due to the symmetry, yields $R = C_5$, contradicting $|R| \geq 6$. □

It turns out that if two graphs, R and B, on the same 5-vertex set do not contain $PRB4$, then $|R| + |B| \leq 13$.

Lemma 4.3. *Let R and B be two graphs on the same vertex set $V = \{v, a, b, x, y\}$, such that $PRB4 \nsubseteq R \cup B$. Then $|R| + |B| \leq 13$ and, if $|R| + |B| \geq 12$, $|R| \geq |B|$, then up to the isomorphism one of the following holds (see Figure 6):*

(A) $R \subseteq K_5[V] - \{ab\}$, $B \subseteq T \cup \{ab\}$, *where* $T = K_3[\{v, x, y\}] = \{vx, vy, xy\}$;
(B) $R = K_5[V]$, $B = \{ab, xy\}$;
(C) $R = B = K_4[\{a, b, x, y\}]$, *where* K_4 *is a complete graph on the vertex set* $\{a, b, x, y\}$;
(D) $R = S_5 \cup \{ab, xy\}$, $B = S_5 \cup \{ax, by\}$, *where* $S_5 = \{va, vb, vx, vy\}$.

Proof. Let two graphs, R and B, on the same vertex set $V = \{v, a, b, x, y\}$, with $PRB4 \nsubseteq R \cup B$ be given. Moreover, let $|R| + |B| \geq 12$, $|R| \geq |B|$, and thereby

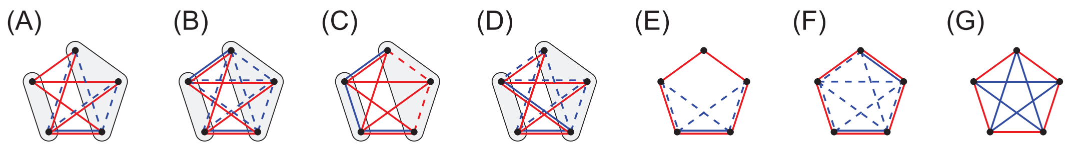


FIGURE 5 The illustration to the proofs of Facts 4.1 and 4.2 [Color figure can be viewed at wileyonlinelibrary.com]

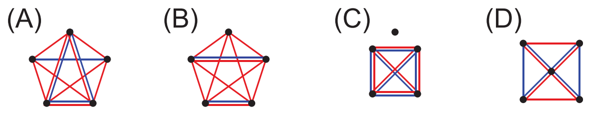


FIGURE 6 All $R \cup B$ on 5 vertices and with $|R| + |B| \geq 12$, such that $PRB4 \nsubseteq R \cup B$ [Color figure can be viewed at wileyonlinelibrary.com]

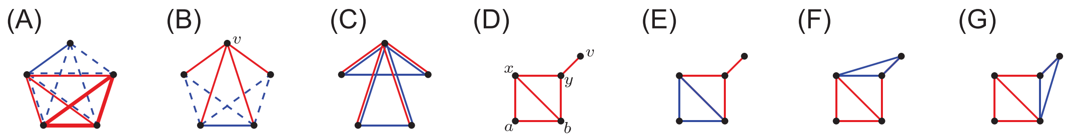


FIGURE 7 The illustration to the proof of Lemma 4.3 [Color figure can be viewed at wileyonlinelibrary.com]

$2 \leq |B| \leq |R| \leq 10$ and $|R| \geq 6$. We will show that one of (A)–(D) occurs. In what follows we assume that $B \nsubseteq M_2$, because otherwise (B) holds.

First observe that $|R| \geq 8$ entails $K_{2,3} \subseteq R$. Then Fact 4.1 combined with $|R| + |B| \geq 12$ tells us that $B \subseteq T \cup \{e\}$. Moreover, $B \nsubseteq M_2$ yields $|B \cap T| \geq 2$. But $|R| \geq 8$ and thus there are at least 4 edges of R between $V[T]$ and e. Therefore, to avoid $PRB4 \subseteq R \cup B$, we have $e \notin R$, and hence (A) follows.

Further, for $|R| \leq 7$ we have $|B| \geq 5$ and thus Facts 4.1 and 4.2 yield that R contains neither $K_{2,3}$ nor C_5. If $K_4 \subseteq R$, then to avoid $PRB4$ in $R \cup B$, every edge $e \in B$ with $|e \cap V[K_4]| = 1$ is an isolated edge in B (see Figure 7A), entailing $|B| \leq 4$. Therefore $B \subseteq K_4$ and hence, using again $PRB4 \nsubseteq R \cup B$, also $R \subseteq K_4$, yielding (C).

Now, as every 5-vertex graph with at least 7 edges contains at least one of the graphs, $K_{2,3}$, C_5, or K_4, as a subgraph, we may assume that $|R| \leq 6$ and thereby $|B| = |R| = 6$. First consider $\Delta(R) = 4$ and let $\deg_R(v) = 4$. Note that $P_2 \nsubseteq B[V \setminus \{v\}]$ (see Figure 7B). Thus $|B[V \setminus \{v\}]| \leq 2$, and $|B| = 6$ entails $S_5 \subseteq B$, where $S_5 = \{va, vb, vx, vy\}$, and $B[V \setminus \{v\}] = M_2$ (see Figure 7C). By the symmetry, $R[V \setminus \{v\}] \subseteq M_2$ and, to avoid a copy of $PRB4$, $R \cap B[V \setminus \{v\}] = \emptyset$, yielding (D).

Finally we let $\Delta(R) \leq 3$ and $|R| = |B| = 6$. The only (up to the isomorphism) $\{K_{2,3}, C_5, K_4, S_5\}$-free graph $G = \{ab, by, xy, ax, bx, yv\}$ with six edges on the vertex set V is given in Figure 7D. Observe that any two edges of one of the triangles abx, xyv, or byv given in Figure 7E–G in blue, create, together with R, a copy of $PRB4$. Therefore $|B| \leq 5$, a contradiction. □

Lemma 4.4. *Let R and B be two graphs on the same 5-vertex set, such that $PRB4 \nsubseteq R \cup B$. If $\Delta(R), \Delta(B) \leq 3$, and at least three vertices of both R and B have degree at most 2, then $|R| + |B| \leq 10$.*

Proof. For the sake of contradiction assume that $|R| + |B| \geq 11$ and let $|R| \geq 6$. Owing to the degree restriction we also have $\max\{|R|, |B|\} \leq 6$, so $5 \leq |B| \leq |R| = 6$. There are exactly two 5-vertex graphs with the degree sequence $(2, 2, 2, 3, 3)$: a pentagon C_5 with one diagonal, and $K_{2,3}$. But then, in view of Facts 4.1 and 4.2, $|B| \leq 4$, a contradiction. □

4.2 | Proof of Lemma 3.1

Let H be a $\mathcal{P}_4$-free 3-graph on a 7-vertex set V and with at least 20 edges. We will show that $H = K_6^{(3)} \cup K_1$, which will end the proof of Lemma 3.1. To this end pick two vertices, $x, y \in V$, with the largest pair degree $\deg_H(x, y) = \Delta_2(H)$ and set $Z = V \setminus \{x, y\}$. We let

$$R = L_H(x)[Z] \quad \text{and} \quad B = L_H(y)[Z]$$

be the link graphs of x and y, respectively, induced on Z. Then,

$$|H| = \deg_H(x,y) + |R| + |B| + |H[Z]| \geq 20. \tag{7}$$

Moreover we have $3 \leq \deg_H(x,y) \leq 5$. Indeed, the upper bound is a trivial consequence of $|Z| = 5$, while the lower bound follows from $\sum_{x,y\in V} \deg_H(x,y) = 3|H| \geq 60$.

We start with estimating the number of edges in the 3-graph $H[Z]$ induced on Z.

Claim 4.5.

(i) If $\deg_H(x,y) = 5$, then $|H[Z]| \leq 2$.
(ii) If $\deg_H(x,y) = 4$, then $|H[Z]| \leq 4$. Moreover, if additionally $|H[Z]| = 4$, then $H[Z] = K_4^{(3)}[N_H(x,y)]$ is a complete 3-graph on the vertex set $N_H(x,y)$.
(iii) If $\deg_H(x,y) = 3$, then $|H[Z]| \leq 6$.

Proof. Clearly, if $|H[Z]| \geq 3$, then there are in $H[Z]$ two edges sharing two vertices, say, abc and bcd. Set z for the unique element of $Z \setminus \{a,b,c,d\}$. Observe that if both z and a are common neighbors of x, y, then the sequence $zxyabcd$ is a minimal 4-path in H (see Figure 8A). As for $\deg_H(x,y) = 5$ each vertex of Z is a common neighbor of x, y, the above observation establishes (i).

For the proof of (ii), instead of looking at edges $e \in H[Z]$, we will look at their *complement edges* $e^c = Z \setminus e$ in Z (e.g., the green 2-edges in Figure 8B are complement edges of the 3-edges $abc, bcd \in H[Z]$ in Figure 8A). In view of this definition, the above observation reads as follows. If there are two adjacent complement edges of $H[Z]$ such that at least one of them is contained in $N_H(x,y)$, then H contains a minimal 4-path (see Figure 8A,B). Therefore if $|H[Z]| \geq 4$, then all complement edges contain the unique vertex of $Z \setminus N_H(x,y)$ (see Figure 8C) and thereby $H[Z] = K_4^{(3)}[N_H(x,y)]$ is a complete 3-graph on the vertex set $N_H(x,y)$.

Finally, to prove (iii) note that (7) together with $\deg_H(x,y) = \Delta_2(H) = 3$ entails

$$17 + 2|H[Z]| \leq |R| + |B| + 3|H[Z]| = \sum_{a,b\in Z} \deg_H(a,b) \leq \binom{5}{2} \cdot \Delta_2(H) = 30.$$ □

Having established Claim 4.5 we proceed with the proof of Lemma 3.1. To this end look at the link graphs R and B, and observe that the $\mathcal{P}_4$-freeness of H entails $PRB4 \not\subseteq R \cup B$ (see Figure 9A).

First assume $\deg_H(x,y) = \Delta_2(H) = 3$. This implies that in each graph, R and B, the vertices $z_1, z_2, z_3 \in N_H(x,y)$ have degree at most 2, while the remaining two vertices of Z have degree at most 3. Hence, by Lemma 4.4, $|R| + |B| \leq 10$. On the other hand, Claim 4.5(iii) together with (7) tell us that $|R| + |B| \geq 11$, a contradiction.

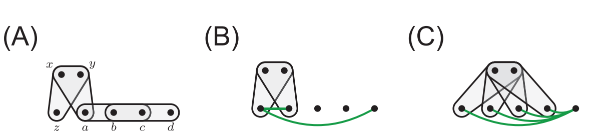


FIGURE 8 The illustration to the proof of Claim 4.5 [Color figure can be viewed at wileyonlinelibrary.com]

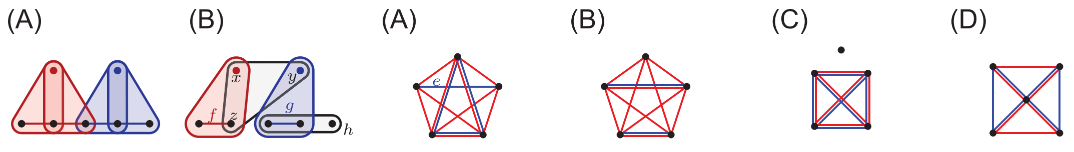


FIGURE 9 The illustration to the proof of Lemma 3.1 [Color figure can be viewed at wileyonlinelibrary.com]

Preparing for the remaining two cases, we make the following observation due to the $\mathcal{P}_4$-freeness of H. Suppose there is a 3-edge $h \in H[Z]$ and two 2-edges, $f \in R$ and $g \in B$ such that $f \cap h = \varnothing$, $f \cap N_H(x, y) \neq \varnothing$, and $g \subset h$. Then, for any vertex $z \in f \cap N_H(x, y)$, 3-edges fx, zxy, yg, h form a minimal 4-path in H, a contradiction (see Figure 9B). Note further that in the above argument one can exchange the graphs R and B.

Next, let $\deg_H(x, y) = 5$. Then (7) and Claim 4.5(i) entails $|R| + |B| \geq 13$. Consequently, in view of Lemma 4.3, $|H[Z]| = 2$ and $|R| + |B| = 13$, and thereby there is a 2-edge $e \in \binom{Z}{2}$ such that, $R = K_5^{(2)}[Z] - e$, $B = K_3^{(2)}[Z \backslash e] \cup e$, because all the other graphs described in (A)–(D) satisfy $|R| + |B| \leq 12$ (see Figure 9A). Now writing $Z = \{a, b, c\} \cup e$, we let $h = ea$, $f = bc$, and $g = e$, which satisfy the assumptions in the previous paragraph and thus yield a contradiction.

Finally, let $\deg_H(x, y) = 4$, and write $N{:}\ = N_H(x, y)$. In view of (7) combined with Claim 4.5(ii), $|H[Z]| \leq 4$ and $|R| + |B| \geq 12$. Then again, Lemma 4.3 tells us that one of (A)–(D) holds. Moreover the condition $\Delta_2(H) = \deg_H(x, y) = 4$ entails that only the unique vertex of $Z \backslash N$ can have degree 4 in R, and thus the cases $R = K_5^{(2)}[Z]$ and $R = K_5^{(2)}[Z] - e$ are excluded. Note that all the remaining 2-graphs $R \cup B$ with $|R| + |B| \geq 12$, described in Lemma 4.3, namely $R = K_5^{(2)}[V] - \{e, e'\}$, $B = T \cup e$, (C), and (D), satisfy $|R| + |B| = 12$, implying that $|H[Z]| = 4$ and thus $H[Z] = K_4^{(3)}[N]$. Moreover, they have the property that every 3-vertex set $h \subset Z$ contains an edge of both 2-graphs R and B (see Figure 9A,C,D), and, as $\deg_H(x, y) = 4$, every edge of $R \cup B$ intersects N. Therefore, if $R \cup B \nsubseteq K_4[N]$, one can take $(f, g) \in (R, B) \cup (B, R)$ with $f \nsubseteq N$ and $h = Z \backslash f \in H[Z]$, $g \subseteq h$, yielding a contradiction with the $\mathcal{P}_4$-freeness of H (see Figure 9B). Thus, we conclude that $R \cup B \subseteq K_4^{(2)}[N]$ and $|R| + |B| = 12$ implies that $R = B = K_4^{(2)}[N]$. Altogether $H = K_6^{(3)} \cup K_1$, as required.

5 | PROOFS OF LEMMAS 3.2 AND 3.3

5.1 | Structure of $\mathcal{P}_4$-free 3-graphs

In this subsection we gather some basic information about the structure of connected $\mathcal{P}_4$-free 3-graphs. We begin by showing that such 3-graphs may contain at most three disjoint edges. To this end, let us make the following observations.

Fact 5.1. For every connected $\mathcal{P}_4$-free 3-graph H the following holds.

(i) If $e_1, e_2 \in H$ are disjoint, then there exists an edge $f \in H$ intersecting both e_1 and e_2.
(ii) If $e_1, e_2 \in H$ are disjoint and $f, h \in H$ are such that $f \cap e_1 \neq \varnothing$, $f \cap e_2 \neq \varnothing$, $h \cap e_1 = \varnothing$, and $h \cap e_2 \neq \varnothing$, then $f \cap h \neq \varnothing$.
(iii) If $e_1, e_2, e_3, f, h \in H$ are such that e_1, e_2, e_3 are pairwise disjoint, $f \cap e_1 \neq \varnothing$, $f \cap e_2 \neq \varnothing$, $f \cap e_3 = \varnothing$, $h \cap e_2 \neq \varnothing$, and $h \cap e_3 \neq \varnothing$, then $h \cap e_1 \neq \varnothing$.
(iv) If $e_1, e_2, e_3 \in H$ are pairwise disjoint, then there exists an edge intersecting all the three edges e_1, e_2, and e_3.

Proof. To prove (i) observe that in a connected 3-graph every pair of disjoint edges, e_1 and e_2, is connected by a minimal path P. If additionally there is no edge in H intersecting both e_1 and e_2, then P consists of at least four edges.

For the proof of (ii) note that otherwise $e_1 f e_2 h$ would form a minimal 4-path in H. Next, to show (iii) observe that $h \cap e_1 = \varnothing$ together with (ii) entails $f \cap h \neq \varnothing$ and, since $f \cap e_3 = \varnothing$, $e_1 f h e_3$ is a minimal 4-path in H, a contradiction.

Finally, to deduce (iv) we apply (i) twice getting two (not necessarily different) edges $f, h \in H$, such that f intersects e_1 and e_2, while h intersects e_2 and e_3. If, additionally, $f \cap e_3 \neq \varnothing$, we are done. Otherwise (iii) yields $h \cap e_1 \neq \varnothing$, which concludes the proof. □

Now we are ready to prove the promised, crucial fact.

Lemma 5.2. *If H is a connected $\mathcal{P}_4$-free 3-graph, then $\nu(H) \leq 3$.*

Proof. Suppose that $\nu(H) \geq 4$ and fix four disjoint edges $e_1, e_2, e_3, e_4 \in H$. Double application of Fact 5.1(iv) entails the existence of two edges, $f, h \in H$, such that f intersects e_1, e_2, e_3, while h intersects e_2, e_3, e_4. Clearly $h \cap e_1 = \varnothing$ and thus, due to Fact 5.1(ii), $f \cap h \neq \varnothing$. But then $e_1 f h e_4$ is a minimal 4-path in H, a contradiction. □

As a preparation towards the proofs of Lemmas 3.2 and 3.3, we now make an attempt to characterize all connected $\mathcal{P}_4$-free 3-graphs with at least two disjoint edges. As an exception, in this section, to distinguish between ordinary graphs (2-graphs) and 3-graphs, we will use notation $\mathcal{F}$, with subscripts, for single 3-graphs rather than families of 3-graphs. (But we keep H unchanged, as it clearly associates itself with hypergraphs).

To this end, recall that a hypergraph $\mathcal{F}$ is *intersecting* if $f \cap f' \neq \varnothing$ for every $f, f' \in \mathcal{F}$. Similarly, a pair $(\mathcal{F}, \mathcal{F}')$ of hypergraphs is called *cross-intersecting*, if for all $f \in \mathcal{F}$, $f' \in \mathcal{F}'$ we have $f \cap f' \neq \varnothing$. It turns out that every connected $\mathcal{P}_4$-free 3-graph H with $\nu(H) \in \{2, 3\}$, can be described as follows.

Lemma 5.3. *Every $\mathcal{P}_4$-free connected 3-graph H with $\nu(H) = 2$ on the set of vertices V, can be partitioned into three edge-disjoint 3-graphs $H = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_{12}$, such that*

(i) $V[\mathcal{F}_1] \cap V[\mathcal{F}_2] = \varnothing$,
(ii) *the 3-graphs $\mathcal{F}_1$ and $\mathcal{F}_2$ are nonempty intersecting families,*
(iii) $\mathcal{F}_{12} \neq \varnothing$,
(iv) *the pair $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{F}_{12})$ is cross-intersecting.*

Lemma 5.4. *Every $\mathcal{P}_4$-free connected 3-graph H with $\nu(H) = 3$ on the set of vertices V, can be partitioned into five edge-disjoint 3-graphs $H = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_{12} \cup \mathcal{F}_{123}$, such that*

(i) *the sets* $V[\mathcal{F}_1]$, $V[\mathcal{F}_2]$, *and* $V[\mathcal{F}_3]$ *are pairwise disjoint, and* $V[\mathcal{F}_{12}] \cap V[\mathcal{F}_3] = \emptyset$,
(ii) *the* 3*-graphs* $\mathcal{F}_1$, $\mathcal{F}_2$, *and* $\mathcal{F}_3$ *are nonempty intersecting families,*
(iii) $\mathcal{F}_{123} \neq \emptyset$,
(iv) *the pairs* $(\mathcal{F}_1 \uplus \mathcal{F}_2 \uplus \mathcal{F}_3 \uplus \mathcal{F}_{12}, \mathcal{F}_{123})$ *and* $(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{F}_{12})$ *are cross-intersecting.*

Proof of Lemmas 5.3 and 5.4. Let H be a given $\mathcal{P}_4$-free connected 3-graph on V, and let $k = \nu(H)$, $k = 2, 3$. Fix a largest matching $M_k = \{e_1, ..., e_k\} \subset H$. Now, for each $I \subseteq [k]$, $\mathcal{F}_I$ is defined to be the set of all edges of H that intersect every e_i, $i \in I$, and none of e_j, $j \in [k]\setminus I$. Clearly $e_i \in \mathcal{F}_{\{i\}}$, $\mathcal{F}_\emptyset = \emptyset$ and

$$H = \biguplus_{I \subseteq [k]} \mathcal{F}_I.$$

For simplicity of notation, we write $\mathcal{F}_{123}$ instead of $\mathcal{F}_{\{1,2,3\}}$, $\mathcal{F}_{12}$ instead of $\mathcal{F}_{\{1,2\}}$, and so forth.

First note that in view of Fact 5.1(iii), for $k = 3$ at most one of $\mathcal{F}_{12}, \mathcal{F}_{13}, \mathcal{F}_{23}$, say $\mathcal{F}_{12}$, is nonempty. Now, if for some vertex $v \in V \setminus (\bigcup_{i \in [k]} e_i)$ there are two edges $f, h \in H$ such that $v \in f \cap h$, $f \in \mathcal{F}_i$ and $h \in \mathcal{F}_j \cup \mathcal{F}_{jk}$, $\{i, j, k\} = \{1, 2, 3\}$, then $e_i f h e_j$ is a minimal 4-path in H. But H is $\mathcal{P}_4$-free and thus the sets $V[\mathcal{F}_1]$, $V[\mathcal{F}_2]$, and $V[\mathcal{F}_3]$ are pairwise disjoint, and $V[\mathcal{F}_{12}] \cap V[\mathcal{F}_3] = \emptyset$, establishing (i). Consequently, as $\nu(H) = k$ and $e_i \in \mathcal{F}_i$ for each $i \in [k]$, every $\mathcal{F}_i$ is a nonempty intersecting family, and thus (ii) follows.

Further, $\mathcal{F}_{12} \neq \emptyset$ and $\mathcal{F}_{123} \neq \emptyset$ result from Fact 5.1(i) and (iv), respectively. Finally, Fact 5.1(ii) tells us that the pairs $(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{F}_{12})$ and $(\mathcal{F}_1 \uplus \mathcal{F}_2 \uplus \mathcal{F}_3 \uplus \mathcal{F}_{12}, \mathcal{F}_{123})$ (for $k = 3$), are cross-intersecting. □

5.2 | Proof of Lemma 3.2

Let H be a $\{\mathcal{P}_4, C_4, M_3\}$-free connected 3-graph on the set of vertices V, $|V| = n \geq 8$, and let $H \nsubseteq S_n^{+1}$, $H \nsubseteq SP_n$, and $H \nsubseteq CB_n$. We are to show that

$$|H| \leq \max\left\{4n - 11, \binom{n-4}{2} + 10\right\}. \tag{8}$$

To prove this observe that because $H \nsubseteq S_n$, if $\nu(H) = 1$, then in view of Theorem 1.3, $|H| \leq 3n - 8 < 4n - 11$, and we are done. Therefore, as H is $M3$-free, we may assume $\nu(H) = 2$ and take a partition

$$H = \mathcal{F}_1 \uplus \mathcal{F}_2 \uplus \mathcal{F}_{12}$$

guaranteed by Lemma 5.3. Recall that both $\mathcal{F}_1$ and $\mathcal{F}_2$ are nonempty intersecting families, $\mathcal{F}_{12} \neq \emptyset$, and the pair $(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{F}_{12})$ is cross-intersecting. For $i = 1, 2$, let $S_i \subseteq V(\mathcal{F}_i)$ be the set of vertices s that lie in all edges of $\mathcal{F}_i$. Clearly, $s_i{:} =|S_i|$ satisfies $0 \leq s_i \leq 3$ and without loss of generality we may assume $0 \leq s_2 \leq s_1 \leq 3$.

Set $V_1 = V[\mathcal{F}_1]$, $V_2 = V \setminus V_1$, and note that $V[\mathcal{F}_2] \subseteq V_2$. A pair p of vertices in V_i is called a 2-cover of $\mathcal{F}_i$ if it intersects every edge of $\mathcal{F}_i$, that is, $p \cap f \neq \emptyset$ holds for all $f \in \mathcal{F}_i$. Denote by $P_i \subseteq \binom{V_i}{2}$ the collection of all 2-covers of $\mathcal{F}_i$. Now (8), and thereby Lemma 3.2, is a straightforward consequence of the following claim.

Claim 5.5.

(i) If $s_1 \geq s_2 \geq 2$, then $|H| \leq 4n - 11$.
(ii) If $s_1 = 3, \quad s_2 = 1, \quad H \nsubseteq S_n^{+1}$, and $H \nsubseteq SP_n$, then $|H| \leq 4n - 11$.
(iii) If $s_1 = 2, \quad s_2 = 1$, and $H \nsubseteq CB_n$, then $|H| \leq \max\left\{4n - 11, \binom{n-4}{2} + 10\right\}$.
(iv) If $s_1 = s_2 = 1$, then $|H| \leq \max\left\{4n - 11, \binom{n-4}{2} + 10\right\}$.
(v) If $s_2 = 0$, then $|H| \leq \max\left\{4n - 11, \binom{n-4}{2} + 10\right\}$.

Proof. Let us start with the proof of (i), that is $s_1 \geq s_2 \geq 2$. To this end, for each $i = 1, 2$ pick an edge $e_i \in \mathcal{F}_i$, and set $W = V \setminus (e_1 \cup e_2), \quad |W| = n - 6$. Then for every $z \in W$,

$$\deg_{\mathcal{F}_1 \cup \mathcal{F}_2}(z) \leq 1 \tag{9}$$

follows from $s_1, s_2 \geq 2$ and $V[\mathcal{F}_1] \cap V[\mathcal{F}_2] = \varnothing$.

Now, let $u, w \in W$ in the case $n = 8$, and $u, w, v \in W$ otherwise, be vertices with the largest degrees in $\mathcal{F}_{12}$, such that

$$\deg_{\mathcal{F}_{12}}(u) \geq \deg_{\mathcal{F}_{12}}(w) \geq \deg_{\mathcal{F}_{12}}(v).$$

We may assume that $\deg_{\mathcal{F}_{12}}(w) \geq 2$. Otherwise, as $\hat{H} = H[e_1 \cup e_2 \cup \{u\}]$ has no isolated vertices, Lemma 3.1 tells us $|\hat{H}| \leq 19$, and by (9) for $n \geq 8$ we have

$$|H| = |\hat{H}| + \sum_{z \in W \setminus \{u\}} \left(\deg_{\mathcal{F}_1 \cup \mathcal{F}_2}(z) + \deg_{\mathcal{F}_{12}}(z)\right) \leq 19 + 2(n - 7) \leq 4n - 11.$$

We contend

$$\deg_H(u) + \deg_H(w) \leq 10 \quad \text{and} \quad \deg_{\mathcal{F}_{12}}(v) \leq 3, \tag{10}$$

which ends the proof. Indeed, observe that the absence of C_4 in H entails $|H[e_1 \cup e_2]| \leq 11$, because, the set of edges of $K_6^{(3)}$ can be partitioned into 10 pairs of disjoint edges, and any two of these pairs form C_4. Therefore (10) combined with (9) tells us

$$|H| = |H[e_1 \cup e_2]| + \deg_H(u) + \deg_H(w) + \sum_{z \in W \setminus \{u,w\}} \left(\deg_{\mathcal{F}_1 \cup \mathcal{F}_2}(z) + \deg_{\mathcal{F}_{12}}(z)\right)$$
$$\leq 4n - 11.$$

To show (10), instead of looking at the degrees of $u, \quad w$, and v it is more convenient for us to look at their link graphs in $\mathcal{F}_{12}$,

$$R = L_{\mathcal{F}_{12}}(u),\ B = L_{\mathcal{F}_{12}}(w),\ \text{and } G = L_{\mathcal{F}_{12}}(v).$$

Because every edge of $\mathcal{F}_{12}$ intersects both e_1 and e_2, actually $R, B, G \subseteq K_{3,3}^{(2)}[e_1 \mathbin{\dot\cup} e_2]$.

We first note that the $\{C_4, \mathcal{P}_4\}$-freeness of H entails some forbidden configurations of edges of $R, \quad B$, and G. In particular, there are no two distinct vertices $x, y \in e_i, \quad i = 1, 2$, such that $\deg_R(x), \deg_B(y) \geq 2$ (see Figure 10A,B, similar with G in place of R or B). This immediately entails that if for some $i = 1, 2$, there are two distinct vertices $x, y \in e_i$ with

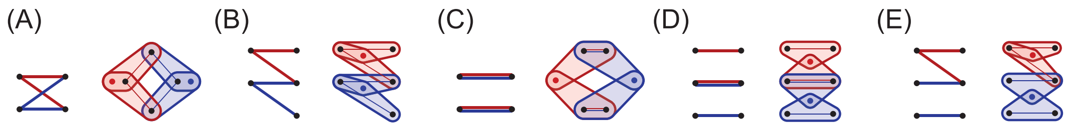


FIGURE 10 Forbidden configurations of edges of R and B [Color figure can be viewed at wileyonlinelibrary.com]

$$\deg_R(x) \geq 2 \quad \text{and} \quad \deg_R(y) \geq 2 \quad \text{then for all} \quad z \in e_i \quad \deg_B(z) \leq 1. \tag{11}$$

In particular, whenever $|R| \geq 6$, then B is a matching and thus $|B| \leq 3$. Moreover, as there are no three disjoint edges in $K_{3,3}^{(2)}[e_1 \cup e_2]$, such that at least two of them are in R and at least two of them are in B (see Figure 10C,D), because $|B| = \deg_{\mathcal{F}_{12}}(w) \geq 2$, we have $|R| \leq 7$ and if $|R| = 7$ then $|B| = 2$. Indeed, otherwise $|R| = 7$ and $|B| = 3$ yields either two disjoint edges in $R \cap B$ (see Figure 10C), or three disjoint edges, two in B and one in R and one edge in R connecting the R-edge with the B-edge, entailing the existence of a minimal 4-path in H (see Figure 10E).

Further, repeated applications of (11) tell us that $|R| = |B| = 5$ is possible only when $R = B = \not{Z}$. But then $R \cap B$ contains two disjoint edges, contradicting C_4-freeness of H (see Figure 10C). For the same reason $|G| \leq 3$. Indeed, otherwise $|R| \geq |B| \geq |G| \geq 4$ and all of these three graphs have the same two vertices of degree larger than one. Thus $R, B, G \subseteq \not{Z}$ (each of them misses at most one edge) and hence the intersection of some two of them contains two disjoint edges, again arriving at a contradiction. Summarizing all these observations so far, we obtain

$$|R| + |B| \leq 9 \text{ and } \deg_{\mathcal{F}_{12}}(v) = |G| \leq 3. \tag{12}$$

Therefore to establish (10) it remains to show that $\deg_H(u) + \deg_H(w) \leq 10$.

To this end, assume for the sake of a contradiction that $\deg_H(u) + \deg_H(w) \geq 11$. Then (12) combined with (9) tells us that

$$\deg_{\mathcal{F}_1 \cup \mathcal{F}_2}(u) = \deg_{\mathcal{F}_1 \cup \mathcal{F}_2}(w) = 1 \text{ and } |R| + |B| = 9.$$

Without loss of generality we may assume that the edge $f \in \mathcal{F}_1 \cup \mathcal{F}_2$ with $w \in f$ belongs to $\mathcal{F}_1$. Recalling that $s_1 \geq 2$ we infer $|e_1 \cap f| = 2$. Now, as every edge of $\mathcal{F}_{12}$ intersects each one of e_1, e_2, f, we actually have $R \subseteq K_{2,3}^{(2)}[(e_1 \cap f) \cup e_2]$ and thus $|R| \leq 6$. Therefore, because $|R| + |B| = 9$ entails $|R| \geq 5$, for $\{x, y\} = e_1 \cap f$ we have $\deg_R(x) \geq 2$ and $\deg_R(y) \geq 2$. Hence (11) tells us $|B| \leq 3$ and if $|R| = 6$, $|B| = 3$, then $R \cap B$ contains two disjoint edges, a contradiction (see Figure 10C).

Before we move to the proof of (ii)–(v) let us show a few simple facts. First note, that for $i = 1, 2$,

$$\deg_{P_i}(v) \leq 3 \text{ for all } v \in V_i \setminus S_i. \tag{13}$$

Indeed, because v is not a 1-cover of $\mathcal{F}_i$, there exists an edge $f \in \mathcal{F}_i$ with $v \notin f$. On the other hand, all 2-covers in P_i intersect f. Hence $N_{P_i}(v) \subseteq f$ and $\deg_{P_i}(v) \leq |f| = 3$ follows. Moreover, as for every edge $h \in \mathcal{F}_i$ we have $|f \setminus h| \leq 2$, one can also deduce that if $v \in h$, then $|N_{P_i}(v) \setminus h| \leq 2$. Thus, in view of (13),

$$|\mathrm{Pi}| \leq \begin{cases} 7 & \text{for} \quad s_i = 0, \\ |V_i| + 3 & \text{for} \quad s_i = 1, \\ 2|V_i| - 2 & \text{for} \quad s_i = 2. \end{cases} \tag{14}$$

To see this, take any edge $h \in \mathcal{F}_i$ and consider neighborhoods in P_i of vertices of h. Clearly, as every 2-cover in P_i intersects h, we have $P_i \subseteq \left\{p \in \binom{V}{2}: p \cap h \neq \varnothing\right\}$. For $s_i = 0$ observe that if there is at most one 2-cover in P_i entirely contained in h, then there are at most six 2-covers in P_i that contain exactly one vertex with h. Similarly, when h contains two 2-covers (they share a vertex), the number of 2-covers in P_i that contain exactly one vertex with h is at most five; and when h contains three 2-covers, this number is at most three. For $s_i = 1$ we let $h = \{s, v, w\}$, where s is the unique 1-cover of $\mathcal{F}_i$. Now, because $\{s, v\}, \{s, w\} \in P_i$, $\deg_{P_i}(s) \leq |V_i| - 1$, $\deg_{P_i}(v) \leq 3$, and $\deg_{P_i}(w) \leq 3$, we actually have $|P_i| \leq (|V_i| - 1) + 2 + 2$. For $s_i = 2$ similar analysis implies $|P_i| \leq (|V_i| - 1) + (|V_i| - 2) + 1$.

Next observe that, as each edge $h \in \mathcal{F}_{12}$ intersects every edge of $\mathcal{F}_1 \cup \mathcal{F}_2$ and $V[\mathcal{F}_1] \cap V[\mathcal{F}_2] = \varnothing$, we have $h = s \cup p$, where $s \in S_i$, $p \in P_j$, $\{i, j\} = \{1, 2\}$. Therefore we can split $\mathcal{F}_{12} = \mathcal{F}_{12}^{\triangleleft} \cup \mathcal{F}_{12}^{\triangleleft}$, where

$$\mathcal{F}_{12}^{\triangleleft} = \{s \cup p \in \mathcal{F}_{12}: s \in S_1, p \in P_2\} \text{ and } \mathcal{F}_{12}^{\triangleleft} = \{p \cup s \in \mathcal{F}_{12}: p \in P_1, s \in S_2\}.$$

Using the absence of C_4 and a member of $\mathcal{P}_4$ in H, one can prove the following fact. Denote by $B_i \subseteq P_i$, $i = 1, 2$, the set of 2-covers of $\mathcal{F}_i$ with at least two neighbors in $\mathcal{F}_{12}$. □

Fact 5.6. For $i = 1, 2$, B_i is an intersecting family. In particular,

$$\begin{aligned} \left|\mathcal{F}_{12}^{\triangleleft}\right| &\leq |P_2| + (s_1 - 1) \cdot |B_2| \leq |P_2| + (s_1 - 1) \cdot \max\{3, \Delta(P_2)\} \text{ and} \\ \left|\mathcal{F}_{12}^{\triangleleft}\right| &\leq |P_1| + (s_2 - 1) \cdot |B_1| \leq |P_1| + (s_2 - 1) \cdot \max\{3, \Delta(P_1)\}. \end{aligned} \tag{15}$$

Proof. Suppose two 2-covers $p, q \in B_i$ of $\mathcal{F}_i$ are disjoint and recall that $s_i \leq 3$. Then H contains either a member of $\mathcal{P}_4$ (see Figure 11A) or C_4 (see Figure 11B), a contradiction. To see (15) recall that the only 2-uniform intersecting families are the triangle and the star, and thus consist of at most $\max\{3, \Delta(P_i)\}$ edges. The inequality $\deg_{\mathcal{F}_{12}}(p) \leq s_i$ follows from $N_{\mathcal{F}_{12}}(p) \subseteq S_i$ for every $p \in P_j$, $\{i, j\} = \{1, 2\}$. □

For $|V_i| = 4$, $i = 1, 2$, every pair of vertices of V_i is a 2-cover of $\mathcal{F}_i$ and thereby $P_i = K_4^{(2)}[V_i]$, yielding $|P_i| = 6$ and $\Delta(P_i) = 3$. Thus, in this case (15) reads as,

$$\text{If } |V_i| = 4 \text{ then } |\mathcal{F}_{12}^{\triangleleft}| \leq 3s_2 + 3 \text{ or } |\mathcal{F}_{12}^{\triangleleft}| \leq 3s_1 + 3 \text{ for } i = 1, 2, \text{ respectively}. \tag{16}$$

Now observe that for $|V_i| = 5$, $i = 1, 2$, each 3-edge of $\mathcal{F}_i$ is disjoint from exactly one pair of vertices of V_i (see Figure 11C). Therefore, for all distinct $x, y \in V_i$ either $\{x, y\} \in P_i$ or $V_i \setminus \{x, y\} \in \mathcal{F}_i$, and hence

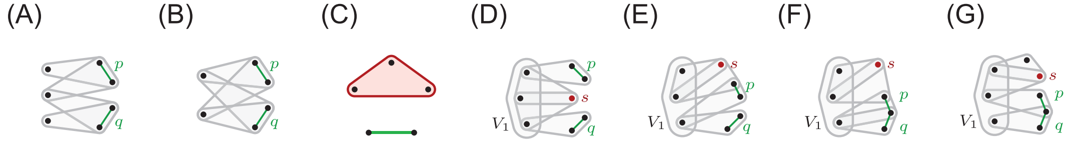


FIGURE 11 The illustration of the proofs of Fact 5.6 and Claim 5.5(ii) [Color figure can be viewed at wileyonlinelibrary.com]

$$\text{If } |V_i| = 5 \text{ then } |\mathcal{F}_i| + |P_i| = \binom{5}{2} = 10. \tag{17}$$

Combining this equality with $\Delta(P_2) \leq 4$ and $|\mathcal{F}_{12}^{\triangleleft}| \leq |P_2| + 4(s_1 - 1)$ ensured by (15), one gets

$$\text{If } |V_2| = 5 \text{ and } s_1 \geq 1 \text{ then } |\mathcal{F}_2| + |\mathcal{F}_{12}^{\triangleleft}| \leq 4s_1 + 6. \tag{18}$$

For the rest of the proof we assume $s_2 \leq 1$ and if $s_2 = 1$, denote by s the unique element of S_2.

Proof of (ii). We let $s_1 = 3$ and thereby $|\mathcal{F}_1| = 1$ and $|V_2| = n - 3 \geq 5$. As $H \nsubseteq S_n^{+1}$ and $H \nsubseteq SP_n$, there are at least two edges $h, h' \in \mathcal{F}_{\triangleleft}^{12}$ disjoint from s. Further, if possible, we choose such $h, \quad h'$ so that $p{:}\ =h \cap V_2$ and $q{:}\ =h' \cap V_2$ are distinct.

First observe, that $p \neq q$. Indeed, otherwise, by our choice of h and h', all edges in $\mathcal{F}_{12}^{\triangleleft} - s$ share the same pair $p \subseteq V_2 \setminus \{s\}$. This implies that $p \in B_2$ and $|\mathcal{F}_{12}^{\triangleleft} - s| \leq 3$. By (13), $s \notin p$ and B_2 is intersecting, the former implies that $|B_2 - p| \leq 2$. Thus, the number of edges in $\mathcal{F}_{12}^{\triangleleft}$ that contain s is at most $(|V_2| - 1) + (s_1 - 1)|B_2 - p| \leq n$, implying that $|\mathcal{F}_{12}^{\triangleleft}| \leq n + 3$. As every edge of $\mathcal{F}_2$ intersects both s and p, one can estimate

$$|\mathcal{F}_2| \leq 2(|V_2| - 3) + 1 = 2n - 11.$$

Putting everything together, we obtain for $n \geq 8$,

$$|H| = |\mathcal{F}_1| + |\mathcal{F}_{12}^{\triangleleft}| + |\mathcal{F}_2| + |\mathcal{F}_{12}^{\triangleleft}| \leq 1 + 3 + (2n - 11) + (n + 3) = 3n - 4 < 4n - 11.$$

Now we proceed by induction on $n \geq 8$ and first consider the base case $n = 8$. For the sake of contradiction suppose $|H| \geq 22$. Since by Fact 5.6 B_2 is intersecting, $|B_2| \leq 4$ and thus,

$$|H| = |\mathcal{F}_1| + |\mathcal{F}_{12}^{\triangleright}| + |\mathcal{F}_2| + |\mathcal{F}_{12}^{\triangleleft}| \overset{(15)}{\leq} 1 + 3 + |\mathcal{F}_2| + |P_2| + (s_1 - 1) \cdot |B_2| \overset{(17)}{=} 14 + 2|B_2| \leq 22,$$

where we used $|\mathcal{F}_{12}^{\triangleleft}| \leq 3$. Therefore the equalities go through meaning $|\mathcal{F}_{12}^{\triangleleft}| = 3, \quad |B_2| = 4$, and $\deg_{\mathcal{F}_{12}}(r) = 3$ for every $r \in B_2$. This, in turn, entails that the link graph of s in $\mathcal{F}_{12}$ is a complete bipartite graph $K_{3,4}^{(2)}[V_1 \cup V_2 \setminus \{s\}]$ and B_2 is a star with center s. But then, as $p \neq q$, no matter where the edges $h, h' \in \mathcal{F}_{12}^{\triangleleft} - s$ are, H contains a minimal 4-path, a contradiction (see Figure 11D–G).

Next suppose $n \geq 9$ and we shall find a vertex of degree at most four so that we could apply induction and conclude the proof. If there are two edges $f_1, f_2 \in \mathcal{F}_2$ with $f_1 \cap f_2 = \{s\}$, set $U := f_1 \cup f_2$. Otherwise, $\mathcal{F}_2 = K_4^{(3)} - e$ and we define $U{:=}V[\mathcal{F}_2]$. Clearly $|U| \leq 5$ and thus we can take a vertex $v \in V_2 \setminus U$. Now every 2-cover $p \in P_2 - s$ of $\mathcal{F}_2$ is entirely contained in U and therefore the only neighbor in P_2 of v is s, yielding $\deg_{\mathcal{F}_{12}^{\triangleleft}}(v) = |N_{\mathcal{F}_{12}^{\triangleleft}}(sv)| \leq |S_1| = 3$. Moreover, because every edge $f \in \mathcal{F}_2$ contains s and intersects both 2-covers $p, q \in P_2 - s$, we have $\deg_{\mathcal{F}_2}(v) \leq 1$. As $\deg_{\mathcal{F}_{12}^{\triangleleft}}(v) = \deg_{\mathcal{F}_1}(v) = 0$, altogether we obtain $\deg_H(v) \leq 4$ and we are done.

Proof of (iii). Let $s_1 = 2$ and $s_2 = 1$, yielding $|V_1| \geq 4, \quad |V_2| \geq 4, \quad |\mathcal{F}_1| \leq |V_1| - 2$, and $|\mathcal{F}_2| \leq \binom{|V_2|-1}{2}$. Moreover, in view of (14) one gets $|\mathcal{F}_{12}^{\triangleleft}| \leq |P_1| \cdot s_2 = |P_1| \leq 2|V_1| - 2$. Now, if there exists an edge $h \in \mathcal{F}_{12}^{\triangleleft}$ disjoint from S_2, then because each edge of $\mathcal{F}_2$ contains s and intersects h, we have $|\mathcal{F}_2| \leq 2(|V_2| - 3) + 1 = 2|V_2| - 5$. Further, applying (14) combined together with (15) yields,

$$|\mathcal{F}_{12}^{\triangleleft}| \leq |P_2| + (s_1 - 1) \cdot \max\{3, \Delta(P_2)\} \leq (|V_2| + 3) + (|V_2| - 1) = 2|V_2| + 2.$$

Summarizing,

$$|H| = |\mathcal{F}_1| + |\mathcal{F}_{12}^{\triangleleft}| + |\mathcal{F}_2| + |\mathcal{F}_{12}^{\triangleleft}| \leq (|V_1| - 2) + (2|V_1| - 2) + (2|V_2| - 5) + (2|V_2| + 2)$$
$$= 4n - |V_1| - 7 \leq 4n - 11.$$

Otherwise s is contained in all edges of $\mathcal{F}_{12}^{\triangleleft}$ (so in fact in all edges of $\mathcal{F}_{12} \cup \mathcal{F}_2$) and thus $|\mathcal{F}_{12}^{\triangleleft}| \leq 2(|V_2| - 1)$. Because $H \nsubseteq CB_n$, we have $|V_1| \geq 5$, entailing $|V_2| \leq n - 5$. Then,

$$|H| \leq (|V_1| - 2) + (2|V_1| - 2) + \binom{|V_2| - 1}{2} + 2(|V_2| - 1) = \binom{|V_2| - 2}{2} + 3n - 8$$
$$\leq \binom{n-4}{2} + 10.$$

Before we proceed observe that for $\{i, j\} = \{1, 2\}$ and each $s' \in S_j$, $H[V_i \cup \{s'\}]$ is an intersecting family. Indeed, this follows from that $\mathcal{F}_i = H[V_i]$ is intersecting, the pair $(\mathcal{F}_i, \mathcal{F}_{12})$ is cross-intersecting, and each edge $h \in H[V_i \cup \{s'\}]$ with $s' \in h$ is in $\mathcal{F}_{12}$. Therefore the celebrated Erdős–Ko–Rado theorem [3] tells us, that for $|V_i| \geq 5$,

$$|H[V_i \cup \{s'\}]| \leq \binom{|V_i|}{2}. \tag{19}$$

Moreover, if there is an edge $h \in H[V_i \cup \{s'\}]$ such that $h \cap S_i = \varnothing$, then for $|V_i| \geq 5$,

$$|H[V_i \cup \{s'\}]| \leq 3|V_i| - 5. \tag{20}$$

For $|V_i| = 5$ the above bound follows from (19), whereas for $|V_i| \geq 6$ one can use Hilton–Milner theorem (Theorem 1.3), as $\mathcal{F}_i \cup \mathcal{F}_{12}[V_i \cup \{s'\}]$ is a nontrivial intersecting family. This is because only vertices of S_i belong to all edges of $\mathcal{F}_i$ and $\mathcal{F}_i \neq \varnothing$, but $h \cap S_i = \varnothing$.

Proof of (iv). We let $s_1 = s_2 = 1$, which entails $|V_i| \geq 4$. Observe, that

$$H[V_1 \cup S_2] = \mathcal{F}_1 \cup \mathcal{F}_{12}^{\triangleleft} \text{ and } H[V_2 \cup S_1] = \mathcal{F}_2 \cup \mathcal{F}_{12}^{\triangleleft}.$$

Therefore, as clearly for $|V_i| = 4$ we have $|H[V_i \cup S_j]| \leq \binom{5}{3} = 10$, $\{i, j\} = \{1, 2\}$, in view of (19),

$$|H| \leq \max\left\{10, \binom{|V_1|}{2}\right\} + \max\left\{10, \binom{|V_2|}{2}\right\} \leq \binom{n-4}{2} + 10,$$

for $n \geq 9$,[1] whereas for $n = 8$ one gets $|H| \leq 10 + 10 \leq 4\cdot 8 - 11$.

Proof of (v). Let $s_2 = 0$ and thereby $\mathcal{F}_{12}^{\triangleleft} = \varnothing$ yielding $H = \mathcal{F}_1 \mathbin{\dot\cup} \mathcal{F}_2 \mathbin{\dot\cup} \mathcal{F}_{12}^{\triangleleft}$. Moreover, $|V_2| \geq 4$ and since $\mathcal{F}_{12}^{\triangleleft} = \mathcal{F}_{12} \neq \varnothing$, $s_1 \geq 1$, which implies $|\mathcal{F}_1| \leq \binom{|V_1|-1}{2}$. If $|V_2| = 4$, then $|V_1| = n - 4 \geq 4$ entailing $s_1 \leq 2$, and therefore (16) tells us $|\mathcal{F}_{12}^{\triangleleft}| \leq 9$. Thus, for $n \geq 8$ we have

$$|H| = |\mathcal{F}_1| + |\mathcal{F}_2| + \left|\mathcal{F}_{12}^{\triangleleft}\right| \leq \binom{n-5}{2} + 4 + 9 \leq \binom{n-4}{2} + 10.$$

Now, let $|V_2| \geq 5$, and pick any $s' \in S_1$. In view of $S_2 = \varnothing$, (20) tells us, $|H[V_2 \cup \{s'\}]| \leq 3|V_2| - 5$. Moreover, by the definition of B_2, (13), (14), and Fact 5.6 we have $|\mathcal{F}_{12}^{\triangleleft} - s'| \leq |P_2| + |B_2| \leq 7 + 3 = 10$. Summarizing,

[1] In particular, when $|V_1|, |V_2| \geq 5$, the inequality can be checked using $|V_1||V_2| \geq 4(n-5) \geq \binom{n}{2} - \binom{n-4}{2} - 10$.

$$|H| \le \binom{|V_1|-1}{2} + (3|V_2|-5) + 10 \le \max\left\{4n-11, \binom{n-4}{2} + 10\right\},$$

where $|H| \le 4n - 11$ can be checked for $3 \le |V_1| \le 5$, and $|H| \le \binom{n-4}{2} + 10$ for $|V_1| \ge 6$.[2]

5.3 | Proof of Lemma 3.3

Let H be a connected $\mathcal{P}_4$-free 3-graph on the set of vertices V, $|V| = n \ge 9$, with $\nu(H) = 3$. Lemma 3.3 follows from

$$|H| \le \binom{n-4}{2} + 11, \tag{21}$$

with the equality achieved if and only if H is the balloon B_n. To prove this inequality we let

$$H = \mathcal{F}_1 \uplus \mathcal{F}_2 \uplus \mathcal{F}_3 \uplus \mathcal{F}_{12} \uplus \mathcal{F}_{123}$$

to be a partition guaranteed by Lemma 5.4. Set $V_1 = V[\mathcal{F}_1]$, $V_3 = V[\mathcal{F}_3]$, and $V_2 = V \setminus (V_1 \cup V_3)$, and recall

(i) $V[\mathcal{F}_2] \subset V_2$, $V_1 \cap V_3 = \emptyset$, and $V[\mathcal{F}_{12}] \cap V_3 = \emptyset$,
(ii) the 3-graphs $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ are nonempty intersecting families,
(iii) $\mathcal{F}_{123} \ne \emptyset$,
(iv) the pairs $(\mathcal{F}_1 \uplus \mathcal{F}_2 \uplus \mathcal{F}_3 \uplus \mathcal{F}_{12}, \mathcal{F}_{123})$ and $(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{F}_{12})$ are cross-intersecting.

Further, for each $i = 1, 2, 3$ pick an edge $e_i \in \mathcal{F}_i$ and split the set of edges of $\mathcal{F}_{12}$ into two subsets, $\mathcal{F}_{12} = \mathcal{F}_{12}^{\text{in}} \uplus \mathcal{F}_{12}^{\text{out}}$, where

$$\mathcal{F}_{12}^{\text{in}} = \{f \in \mathcal{F}_{12} : f \subset e_1 \cup e_2\} \text{ and } \mathcal{F}_{12}^{\text{out}} = \{f \in \mathcal{F}_{12} : |f \cap e_1| = |f \cap e_2| = 1\}.$$

Because every edge of $\mathcal{F}_{12}$ intersects both e_1 and e_2, we have

$$H = \mathcal{F}_1 \uplus \mathcal{F}_2 \uplus \mathcal{F}_3 \uplus \mathcal{F}_{12}^{\text{in}} \uplus \mathcal{F}_{12}^{\text{out}} \uplus \mathcal{F}_{123}. \tag{22}$$

The proof of (21) mainly relies on two technical claims enabling us to bound the number of edges in $\mathcal{F}_{12} \cup \mathcal{F}_{123}$. In the first of them we estimate the size of $\mathcal{F}_{123} \cup \mathcal{F}_{12}^{\text{in}}$.

Claim 5.7. $|\mathcal{F}_{123}| + |\mathcal{F}_{12}^{\text{in}}| \le 18$. Moreover, if $|\mathcal{F}_{123}| + |\mathcal{F}_{12}^{\text{in}}| = 18$ then $\mathcal{F}_{123} \cup \mathcal{F}_{12}^{\text{in}}$ is a star.

Proof. As every edge $h \in \mathcal{F}_{123}$ intersects each one of e_1, e_2, and e_3, we trivially have $|\mathcal{F}_{123}| \le 27$, but this estimate can be improved. Let $G \subseteq K_{3,3}^{(2)}[e_1 \cup e_2]$ be an auxiliary bipartite graph with vertex classes e_1 and e_2, consisting of all pairs $\{u, v\} \in e_1 \times e_2$ for which there exists a vertex $w \in e_3$ such that $uvw \in \mathcal{F}_{123}$. It turns out that the number of edges in $\mathcal{F}_{123}$ can exceed $|G|$ only by at most 6,

$$|\mathcal{F}_{123}| \le |G| + 6. \tag{23}$$

[2]The inequality holds for $|V_2| = 5$. For $|V_2| \ge 6$, we have $\binom{n-4}{2} - \binom{|V_1|-1}{2} = \binom{n-4}{2} - \binom{n-|V_2|-1}{2} \ge 3(n-6) \ge 3|V_2|$.

Indeed, clearly any edge of G can be extended to at most 3 edges of $\mathcal{F}_{123}$ (see Figure 12A). However, due to the $\mathcal{P}_4$-freeness of H, there can be no two disjoint edges $f_1, f_2 \in G$ and three different vertices $w_1, w_2, w_3 \in e_3$, such that $f_1w_1, f_1w_2, f_2w_2, f_2w_3$ are all edges in $\mathcal{F}_{123}$, as they would form a minimal 4-path in H (see Figure 12B). Similarly, there are no disjoint edges $f_1, f_2, f_3 \in G$ and vertices $w_1, w_2 \in e_3$ with $f_1w_1, f_2w_1, f_2w_2, f_3w_2 \in \mathcal{F}_{123}$ (see Figure 12C). To avoid such structures, any two disjoint edges in G can be extended, in total, to at most 4 edges of $\mathcal{F}_{123}$, and any three disjoint edges of G can be extended, in total, to at most 5 edges of $\mathcal{F}_{123}$. Therefore, to conclude (23) it is enough to observe, that the set of edges of $K_{3,3}^{(2)}$ can be partitioned into three disjoint matchings $M_3^{(2)}$, say $\mathrm{M}R$, $\mathrm{M}G$, $\mathrm{M}B$ (see Figure 12D). Now, for each $i \in \{R, G, B\}$, $G \cap \mathrm{M}_i$ can be extended to at most $|G \cap \mathrm{M}_i| + 2$ edges of $\mathcal{F}_{123}$.

Next, let us note that

$$|G| \geq 4 \text{ entails } |\mathcal{F}_{12}^{\text{in}}| \leq 6. \tag{24}$$

To show this, recall that every edge $f \in \mathcal{F}_{12}^{\text{in}}$ intersects each $uvw \in \mathcal{F}_{123}$ and thereby also every $uv \in G$. As there are only five pairwise non-isomorphic subgraphs of $K_{3,3}^{(2)}$ with four edges (all listed in Figure 13A–E), a simple case analysis enables us to establish (24).

Finally observe that

$$\mathcal{F}_{12}^{\text{in}} \neq \emptyset \text{ entails } |G| \leq 7, \text{ and } |\mathcal{F}_{12}^{\text{in}}| \geq 5 \text{ yields } |G| \leq 5. \tag{25}$$

Indeed, as $(\mathcal{F}_{12}^{\text{in}}2, G)$ is cross-intersecting, the existence of any edge in $\mathcal{F}_{12}^{\text{in}}$ forbids two pairs from G (see Figure 13F). Moreover, among every 5 edges of $\mathcal{F}_{12}^{\text{in}}$ there are two, f_1, f_2, sharing at most one vertex. Therefore, as every edge $g \in G$ intersects both f_1 and f_2, out of all 9 edges of $K_{3,3}^{(2)}$ at least four are forbidden for G, yielding $|G| \leq 5$ (see Figure 13G,H).

Now we are ready to finish the proof of Claim 5.7. To this end assume

$$|\mathcal{F}_{123}| + |\mathcal{F}_{12}^{\text{in}}| \geq 18 \tag{26}$$

and note that, in view of (23), this entails $|G| + |\mathcal{F}_{12}^{\text{in}}| \geq 12$. Combining this estimate with (24) and (25) one can conclude that $|G| \leq 3$. Indeed, as $|G| \leq 9$ we have $\mathcal{F}_{12}^{\text{in}} \neq \emptyset$ and thus $|G| \leq 7$. Next, assuming $|G| \geq 4$ we get $|\mathcal{F}_{12}^{\text{in}}| \leq 6$ and $|G| \leq 5$, implying $|G| + |\mathcal{F}_{12}^{\text{in}}| \leq 11$.

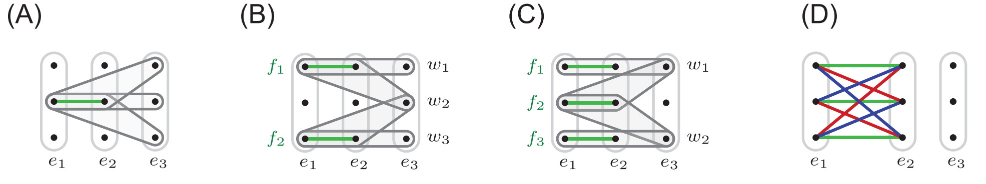


FIGURE 12 Extensions of edges of G and decomposition of $K_{3,3}^{(2)}$ into matchings [Color figure can be viewed at wileyonlinelibrary.com]

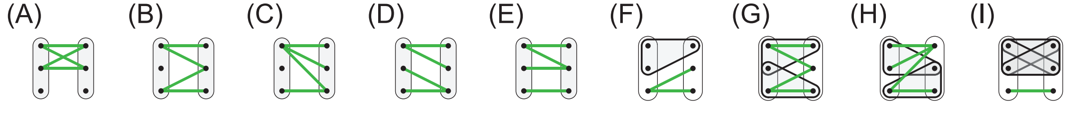


FIGURE 13 All 4-edge subgraphs of $K_{3,3}^{(2)}$ and forbidden edges of G and F_{12}^{in} [Color figure can be viewed at wileyonlinelibrary.com]

To exclude $|G| \leq 2$ let us recall again that$(\mathcal{F}_{12}^{\text{in}}, G)$ is cross-intersecting, and observe that because every edge of G is disjoint from four edges of $\mathcal{F}$in12 (see Figure 13I), $|G| = 1$ results $|\mathcal{F}_{12}^{\text{in}}| \leq 18 - 4 = 14$. Similarly, $|G| = 2$ entails $|\mathcal{F}_{12}^{\text{in}}| \leq 11$. As every edge of G can be extended to at most 3 edges of $\mathcal{F}_{123}$, in both cases $|\mathcal{F}_{123}| + |\mathcal{F}_{12}^{\text{in}}| \leq 17$. But this, together with $G \neq \emptyset$ guaranteed by (iii), contradicts (26). Thus, $|G| = 3$ and thereby $|\mathcal{F}_{12}^{\text{in}}| \geq 9$. A quick inspection shows that this is possible only when both G and $\mathcal{F}_{12}^{\text{in}}$ are stars with the same center. □

Our next goal is to bound the number of edges in $\mathcal{F}_{12}^{\text{out}}$.

Claim 5.8. If there exists a vertex $v \in V \backslash (e_1 \cup e_2 \cup e_3)$ with $\deg_{\mathcal{F}_{12}}(v) \geq 4$, then $|H| \leq \binom{n-4}{2} + 10$.

Proof. We let $v \in V \backslash (e_1 \cup e_2 \cup e_3)$ to be a vertex with $\deg_{\mathcal{F}_{12}}(v) \geq 4$. Split the vertex set $V = R \uplus S \uplus V_3$, where

$$R = e_1 \cup e_2 \cup \{v\}, \; S = V \backslash (R \cup V_3),$$

and $R \cap V_3 = \emptyset$ follows from (i).

We begin by proving, that every vertex $w \in S$ satisfies

$$\deg_H(w) \leq 7. \tag{27}$$

Indeed, we let $h \in \mathcal{F}_{123}$ to be an edge guaranteed by (iii), and set $\{x_i\} = h \cap e_i, \quad i = 1, 2, 3$. Now (iv) tells us, that every edge $f \in \mathcal{F}_{12}$ intersects h and thus contains at least one of the vertices x_1, x_2. This entails $\deg_{\mathcal{F}_{12}}(w) \leq 5$ (see Figure 14A), and therefore it remains to show that $\deg_{\mathcal{F}_1 \cup \mathcal{F}_2}(w) \leq 2$.

For this purpose, recall that in view of (i) every vertex $w \in S$ can have positive degree only in one of the graphs $\mathcal{F}_1, \mathcal{F}_2$, say $\mathcal{F}_1$. Next observe, that there exists an edge $f \in \mathcal{F}_{12}$ disjoint from $\{x_1, w\}$, because only three out of at least four edges of $\mathcal{F}_{12}$ containing v can be incident to x_1 (see Figure 14B). Now, repeated application of (iv) tells us that every edge $e \in \mathcal{F}_1$ intersects both h and f, and thereby contains x_1 and one of two vertices of $f \backslash e_2$. Clearly w is contained in at most two of such edges (see Figure 14C).

Further we claim that

$$|\mathcal{F}_{123}| \leq 3, \tag{28}$$

because every edge $h' \in \mathcal{F}_{123}$ contains both x_1 and x_2. Indeed, if not, let $h' = x_1'x_2'x_3'$ and say $x_2 \neq x_2'$. Then, as in view of (iv), every edge of $\mathcal{F}_{12}$ intersects both h and h', either $N_{\mathcal{F}12}(v) \subseteq \{x_1\} \times e_2$ if $x_1 = x_1'$, or $N_{\mathcal{F}_{12}}(v) \subseteq \left\{x_1'x_2, x_1x_2'\right\}$ otherwise, contradicting $\deg_{\mathcal{F}_{12}}(v) \geq 4$.

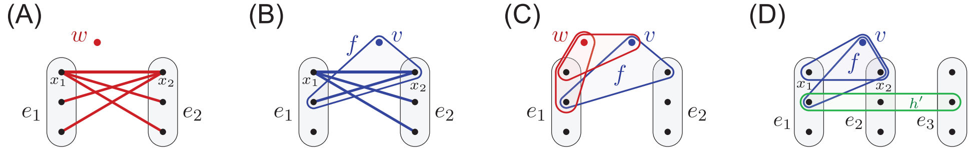


FIGURE 14 Possible neighbors of $w \in S$ in $\mathcal{F}_{12}$ and $\mathcal{F}_1$. The link graphs of w and v are denoted by red and blue 2-edges, respectively [Color figure can be viewed at wileyonlinelibrary.com]

Now we are ready to finish the proof of Claim 5.8. To this end denote $|V_3| = t$, and thereby $|S| = n - 7 - t$, as clearly $|R| = 7$. Moreover, we let

$$H_S = \{h \in H : h \cap S \neq \varnothing\},$$

and observe that (i) entails

$$H = \mathcal{F}_3 \cup \mathcal{F}_{123} \cup H[R] \cup H_S.$$

Next note, that Lemma 3.1 combined with $\mathcal{P}_4$-freeness of H tells us $|H[R]| \leq 19$, and (iv) yields $x_3 \in f$ for each $f \in \mathcal{F}_3$, causing $|\mathcal{F}_3| \leq \binom{t-1}{2}$. Altogether, in view of (27) and (28), for $n \geq 8$,

$$|H| = |\mathcal{F}_3| + |\mathcal{F}_{123}| + |H[R]| + |H_S| \leq \binom{t-1}{2} + 3 + 19 + 7(n - 7 - t)$$
$$\leq \binom{n-4}{2} + 10,$$

as the left-hand side of the last inequality achieves its maximum for either $t = 3$ or $t = n - 6$. □

Having established the above claims we proceed with the proof of (21). To this end, recall that (iii) combined with (iv) entail, that for each $i = 1, 2, 3,\quad \mathcal{F}_i$ is a star, and thus $|\mathcal{F}_i| \leq \binom{|V_i|-1}{2}$. Therefore, in view of (i), by simple optimization,

$$|\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \leq \binom{n-7}{2} + 2,$$

with the equality achieved if and only if one of the 3-graphs $\mathcal{F}_i,\quad i = 1, 2, 3$ is a full star on $n - 6$ vertices, whereas two remaining 3-graphs each consists of a single edge. Further, we may assume that each vertex $v \in V \setminus (e_1 \cup e_2 \cup e_3)$ satisfies $\deg_{\mathcal{F}_{12}}(v) \leq 3$, and thereby $|\mathcal{F}_{12}^{\text{out}}| \leq 3(n - 9)$, since otherwise Claim 5.8 tells us that $|H| \leq \binom{n-4}{2} + 10$, and (21) follows without the equality. Combining these observations together with (22) and Claim 5.7 one gets

$$|H| = (|\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3|) + \left(|\mathcal{F}_{123}| + \left|\mathcal{F}_{12}^{\text{in}}\right|\right) + \left|\mathcal{F}_{12}^{\text{out}}\right| \leq \binom{n-7}{2} + 2 + 18 + 3(n - 9)$$
$$= \binom{n-4}{2} + 11,$$

as required. It remains to show that the equality in the above bound is achieved if and only if H is a balloon.

Indeed, clearly if $|H| = \binom{n-4}{2} + 11$, then equalities in the above formula go through. In particular, if $n = 9$, then $\mathcal{F}_{12}^{\text{out}} = \varnothing,\quad \mathcal{F}_i,\quad i = 1, 2, 3,$ is a single edge and $\mathcal{F}_{123} \cup \mathcal{F}_{12}^{\text{in}}$ is a star with center in $e_1 \cup e_2$. It is easy to see that $H = B_9$.

Now assume $n \geq 10,\quad |\mathcal{F}_{12}^{\text{out}}| = 3(n - 9)$ and thus $|V[\mathcal{F}_{12}]| = n - 3$ yielding, in view of (i), $|V_3| = 3$. Therefore, as $|\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| = \binom{n-7}{2} + 2$, without loss of generality we may assume that $\mathcal{F}_1$ is a full star on $n - 6 \geq 4$ vertices, whereas $|\mathcal{F}_2| = |\mathcal{F}_3| = 1$. Let $c \in e_1$ be the center of $\mathcal{F}_1$. As the pair $(\mathcal{F}_1, \mathcal{F}_{12} \cup \mathcal{F}_{123})$ is cross-intersecting and $\mathcal{F}_{123} \cup \mathcal{F}_{12}^{\text{in}}$ is a star, the center of $\mathcal{F}_{123} \cup \mathcal{F}_{12}^{\text{in}}$ must also be c as for any vertex $v \neq c$, the *full* star $\mathcal{F}_1$ contains an edge not containing v. Finally, since $\mathcal{F}_{123}$ contains all possible 9 edges containing c and the pair $(\mathcal{F}_{123}, \mathcal{F}_{12})$ is cross-intersecting, every edge of $\mathcal{F}_{12}^{\text{out}}$ contains c as well. Altogether $\mathcal{F}_1 \cup \mathcal{F}_{12} \cup \mathcal{F}_{123}$ is a star, whereas $\mathcal{F}_2$ and $\mathcal{F}_3$ are single edges, and thereby $H = B_n$.

6 | PROOFS OF LEMMAS 3.4 AND 3.5

Let H be a connected $\mathcal{P}_4 \cup \{M_3\}$-free 3-graph on the set of n vertices V, $n \geq 8$, such that $C_4 \subseteq H$. Denote by

$$C = \{x_1y_1y_2, x_1z_1z_2, x_2y_1y_2, x_2z_1z_2\}$$

a copy of C_4 contained in H, and set $V[C] = Z = \{x_1, x_2, y_1, y_2, z_1, z_2\}$, $W = V \setminus Z$.

Lemmas 3.4 and 3.5 are straightforward consequences of the following two lemmas.

Lemma 6.1. *If there exist two vertices $u, w \in W$ with degree in H at least 5, and moreover either*

(i) $|H[Z \cup \{u, w\}]| \geq 22$ *or*
(ii) *there is a further vertex $v \in W \setminus \{u, w\}$ with $\deg_H(v) \geq 5$,*

then $H \subseteq SP_n$.

Lemma 6.2. *If there exist two vertices $u, w \in W$, such that*

(i) $|H[Z \cup \{u, w\}]| \geq 22$ *and*
(ii) $\deg_H(w) \leq 4$,

then $H \subseteq SK_n$.

Indeed, assume first that there are two vertices $u, w \in W$, such that $|H \cup \{u, w\}| \geq 22$. Then, either $\deg_H(u), \deg_H(w) \geq 5$ and thus, in view of Lemma 6.1, $H \subseteq SP_n$, or the degree in H of one of these vertices, say w, is at most 4. Then Lemma 6.2 tells us that $H \subseteq SK_n$.

So let for every pair of vertices $u, w \in W$, $|H[Z \cup \{u, w\}]| \leq 21$. If there are three vertices $u, w, v \in W$, with the degree in H at least 5, then due to Lemma 6.1, $H \subseteq SP_n$. Otherwise choose $u, w \in W$ in such a way, that for all $v \in W \setminus \{u, w\}$, $\deg_H(v) \leq 4$. Then,

$$|H| = |H[Z \cup \{u, w\}]| + \sum_{v \in W \setminus \{u,w\}} \deg_H(v) \leq 4n - 11.$$

Altogether, either $H \subseteq SP_n$, $H \subseteq SK_n$, or $|H| \leq 4n - 11$. Now, as $SK_n \nsubseteq SP_n$, $|SP_n| = 5n - 18$, and $|SK_n| = 4n - 10$, Lemmas 3.4 and 3.5 follows from

$$5n - 18 \geq 4n - 10 > 4n - 11,$$

for $n \geq 8$, with the equality only for $n = 8$.

6.1 | Preliminaries

We begin with a series of technical results which will be helpful in the proofs of Lemmas 6.1 and 6.2. Throughout, we denote by u and w arbitrary vertices of W. The $\mathcal{P}_4 \cup \{M_3\}$-freeness of H implies that for all edges $h \in H$

$$|h \cap Z| \geq 2, \quad h \cap Z \neq \{y_1, y_2\}, \quad h \cap Z \neq \{z_1, z_2\}. \tag{29}$$

Let us partition H into four edge-disjoint sub-3-graphs,

$$H = H_Z \cup H^0 \cup H^1 \cup H^2,$$

where $H_Z = H[Z]$ and, for $i = 0, 1, 2$,

$$H^i = \{h \in H \setminus H_Z : |h \cap \{x_1, x_2\}| = i\}.$$

The first inequality in (29) implies that the link graph $L_H(w)$ of every vertex $w \in W$ is entirely contained in $\binom{Z}{2}$. Moreover, the above partition of H induces a corresponding partition of $L_H(w)$,

$$L_H(w) = H^0(w) \cup H^1(w) \cup H^2(w),$$

where $H^i(w) = L_{H^i}(w) = \{e \in L_H(w) : |e \cap \{x_1, x_2\}| = i\}$. Observe, that

$$|H^0(w)| \leq 4, \ |H^1(w)| \leq 8, \ \text{and} \ |H^2(w)| \leq 1, \tag{30}$$

where the first inequality holds, because in view of (29), $\{y_1, y_2\}, \{z_1, z_2\} \notin L_H(w)$, and thus $H^0(w)$ is a subgraph of the 4-cycle $y_1 z_1 y_2 z_2$.

Our first result describes the structure of $H^1(w)$ and, as a consequence, halves the upper bound on $|H^1(w)|$.

Fact 6.3. For every $w \in W$, $H^1(w)$ is either a star (with the center at x_1 or x_2) or a subgraph of one of the 4-cycles: $C_y = x_1 y_1 x_2 y_2$ or $C_z = x_1 z_1 x_2 z_2$. In particular, $|H^1(w)| \leq 4$.

Proof. If there were two disjoint edges in $H^1(w)$, one contained in C_y and the other in C_z, say $\{x_1, y_2\}$ and $\{x_2, z_1\}$, then $y_1 y_2 x_1 w x_2 z_1 z_2$ would form a minimal 4-path in H, a contradiction (see Figure 15). So, either all edges of $H^1(w)$ are contained in one of the cycles, C_y or C_z, or they form a star. □

It is convenient to break the 3-graph H_Z into three further sub-3-graphs,

$$H_Z = H_Z^0 \cup H_Z^1 \cup H_Z^2, \ \text{where} \ H_Z^i = \{h \in H_Z : |h \cap \{x_1, x_2\}| = i\}, \ i = 0, 1, 2.$$

Note that $C \subseteq H_Z^1$, $\quad |H_Z^0| \leq \binom{4}{3} = 4, \quad |H_Z^1| \leq 2\binom{4}{2} = 12$, and $|H_Z^2| \leq \binom{4}{1} = 4$.

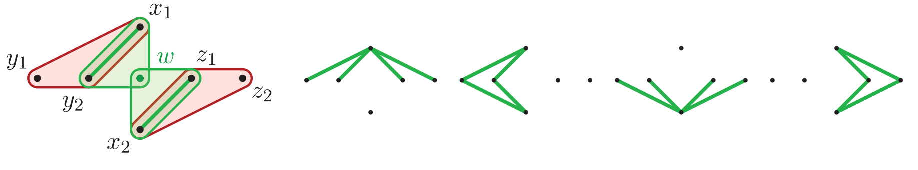


FIGURE 15 A minimal 4-path $y_1 y_2 x_1 w x_2 z_1 z_2$ in H and all possible edges of link graphs $H^1(w)$ [Color figure can be viewed at wileyonlinelibrary.com]

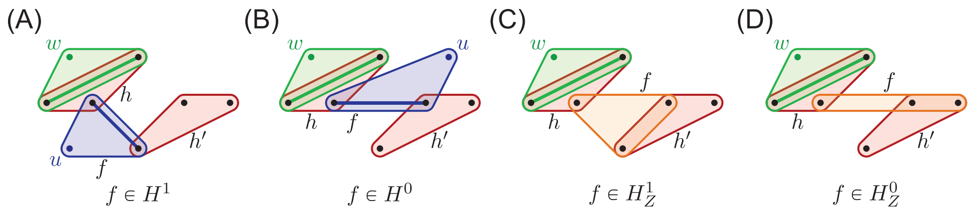


FIGURE 16 Illustration to the proof of Fact 6.4 [Color figure can be viewed at wileyonlinelibrary.com]

The next result lists several basic observations on the above-defined subgraphs, all stemming from the $\mathcal{P}_4$-freeness of H.

Fact 6.4. Let $h, h' \in C$ be disjoint and let, for some vertex $w \in W$, an edge $e \in H^1(w)$ be contained in h. Then there is no edge $f \in H - w$ disjoint from e and intersecting both h and h'. Consequently, for any two distinct vertices $u, w \in W$, the following properties hold,

- (i) if $e \in H^1(u)$ and $e' \in H^1(w)$ are disjoint, then there exist disjoint $h, h' \in C$ such that $e \subset h$ and $e' \subset h'$;
- (ii) the pair of 2-graphs $(H^0(u), H^1(w))$ is cross-intersecting;
- (iii) if $e \in H^1(w)$ and $f \in H^0_Z \cup H^1_Z$, $\quad f \notin C$, then $e \cap f \neq \varnothing$;
- (iv) if $H^1(w) \neq \varnothing$ then $|H^1_Z| \leq 10$;
- (v) if $|H^1(u) \cup H^1(w)| \geq 2$ then $|H^1_Z| \leq 9$ and $|H_Z| \leq 15$;
- (vi) if $|H^1(u) \cup H^1(w)| \geq 3$ then $|H^1_Z| \leq 8$ and $|H_Z| \leq 13$;
- (vii) if $|H^1(w)| = 4$ then $|H_Z| \leq 12$;
- (viii) if $|H^1(u) \cup H^1(w)| \geq 7$ then $|H_Z| \leq 8$;
- (ix) if $H^1(w)$ is a star with four edges and the center x_1 or x_2, then $H \subseteq SP_n$.

Proof. Suppose that $h, h' \in C$, $\quad w \in W$, $\quad e \in H^1(w)$, and $f \in H - w$ are such that $h \cap h' = \varnothing$, $\quad e \subset h$, and $f \cap h = h \backslash e$ and $f \cap h' \neq \varnothing$. Then, regardless of the location of f, the 3-edges we, h, f, and h' form a minimal 4-path in H (see Figure 16), contradicting the $\mathcal{P}_4$-freeness of H. So the main statement is proved, and consequently, (i)–(iii) follow. Indeed, if (i), (ii), or (iii) were not true, then we would be looking at the forbidden configurations in Figure 16A,B or 16C,D, respectively.

In turn, (iii) implies (iv)–(viii). Indeed, (iv) follows from the bound $|H^1_Z| \leq 12$ as, in view of (iii), $H^1(w) \neq \varnothing$ excludes two edges from H^1_Z. Similarly, in (v), considering five different cases with respect to the location of the two edges of $H^1(u) \cup H^1(w)$, we may exclude (by applying [iii]) at least 3 edges of H^1_Z and at least 5 edges of $H^0_Z \cup H^1_Z$. By the same token, in (vi), we exclude at least 4 edges of H^1_Z and at least 7 edges of $H^0_Z \cup H^1_Z$. For the proof of (vii), recall that Fact 6.3 tells us that $H^1(w)$ is either a 4-arm star or one of the cycles C_y or C_z. In both cases, via (iii), it wipes out at least 4 edges of H^1_Z and at least 8 edges of $H^0_Z \cup H^1_Z$. We leave case (viii) for the Reader.

Finally, to prove (ix), assume, without loss of generality, that $H^1(w)$ is a 4-edge star with the center x_1. Now observe that by (i)–(iii) every edge of H, except for $x_2y_1y_2$ and $x_2z_1z_2$ (which form $P = P_2^{(3)}$ disjoint from $\{x_1\}$), contains both x_1 and a member of $V[P] = \{y_1, y_2, x_2, z_1, z_2\}$, entailing $H \subseteq SP_n$. Indeed, (ii) yields that $H^0(u) = \varnothing$ and

$H^1(u) \subseteq H^1(w)$ holds by (i). Moreover (iii) tells us that $H_Z^0 = \varnothing$ and whenever $f \in H_Z^1 \setminus C, \quad x_1 \in f$. □

Corollary 6.5. *For all distinct $u, w \in W$,*

(i) $H^1(u) \neq \varnothing \Rightarrow |H^0(w)| \leq 2$;
(ii) $H^0(u) \neq \varnothing \Rightarrow |H^1(w)| \leq 2$.

Proof. Observe that for any edge $e \in H^1(u)$, there exist at most two edges in $H^0(w)$ which intersect e. Similarly, by Fact 6.3, for any edge $e \in H^0(u)$ there are at most two edges in $H^1(w)$ sharing a vertex with e. Consequently, by Fact 6.4(ii), both assertions follow. □

Corollary 6.6. *If there is a vertex $u \in W$ with $\deg_H(u) \geq 6$, then the degree of every vertex $w \in W \setminus \{u\}$ is at most 3. Moreover, if additionally $\deg_H(w) = 3$, then $|H^2(w)| = 1$, that is, $wx_1x_2 \in H$.*

Proof. Let $\deg_H(u) \geq 6$ and let $w \in W \setminus \{u\}$. By (30) and Fact 6.3, both sets, $H^0(u)$ and $H^1(u)$, must be nonempty and at least one of them of size at least three, say $|H^1(u)| \geq 3$. But then, by Corollary 6.5(ii), $|H^1(w)| \leq 2$ and $H^0(w) = \varnothing$. Hence,

$$\deg_H(w) = |L_H(w)| = |H^0(w)| + |H^1(w)| + |H^2(w)| \leq 0 + 2 + 1 = 3$$

and if $\deg_H(w) = 3$, then $|H^2(w)| = 1$. □

Fact 6.7. If $ux_1x_2, wx_1x_2 \in H^2$ and $e \in H^0(w)$, then there is no $f \in H[V \setminus \{x_1, x_2, u, w\}]$ with $f \cap e \neq \varnothing$. It follows that $H_Z^0 = \varnothing$. Moreover, if $|H^0(w)| \geq 2$, then for every $v \in W \setminus \{u, w\}$, we have $H^0(v) = \varnothing$.

Proof. To prove the first statement, it is enough to observe that whenever $ux_1x_2, x_1x_2w \in H, \quad e \in H^0(w)$, and $f \in H[V \setminus \{x_1, x_2, u, w\}], \quad f \cap e \neq \varnothing$, then edges $ux_1x_2, \quad x_1x_2w, \quad we$, and f form a minimal 4-path in H. As e uses two of the four vertices of $V\left[H_Z^0\right]$, there is no room for an $f \in H_Z^0$ with $f \cap e = \varnothing$, and so $H_Z^0 = \varnothing$. Furthermore, if $|H^0(w)| \geq 2$ and $e' \in H^0(v)$ then $f = e'v \in H[V \setminus \{x_1, x_2, u, w\}]$ and there exists $e \in H^0(w)$ such that $f \cap e = e' \cap e \neq \varnothing$, a contradiction. □

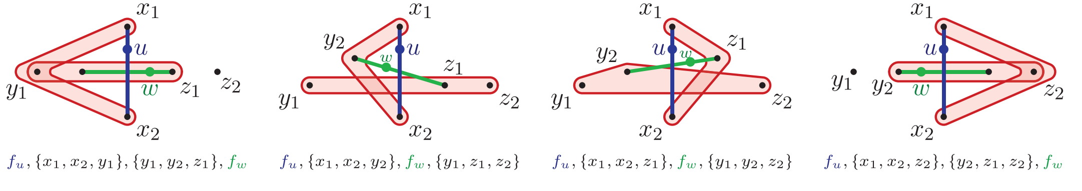


FIGURE 17 A minimal 4-paths with edges f_u (blue) and f_w (green) [Color figure can be viewed at wileyonlinelibrary.com]

Fact 6.8. If $ux_1x_2 \in H^2$ and $H^0(w) \neq \emptyset$, then $|H_Z^0| + |H_Z^2| \leq 4$.

Proof. Let $f_u = ux_1x_2 \in H^2$ and $f_w \in H^0$, $w \in f_w$. Without loss of generality we may assume that $f_w = wy_2z_1$. Observe that $H_Z^0 \cup H_Z^2$ can be partitioned into four pairs of edges,

$$\{x_1x_2y_1, y_1y_2z_1\}, \{x_1x_2y_2, y_1z_1z_2\}, \{x_1x_2z_1, y_1y_2z_2\}, \{x_1x_2z_2, y_2z_1z_2\},$$

such that each of them, together with edges f_u and f_w, forms a minimal 4-path in H (see Figure 17). Consequently, from each of these pairs only one edge may belong to H_Z. □

Fact 6.9. If $ux_1x_2 \in H^2$ and $|H^0(w)| \geq 2$, then $|H_Z^1| \leq 8$, $|H_Z^2| \leq 2$ and $|H_Z| \leq 12$.

Proof. Let $f_u = ux_1x_2 \in H^2$ and $e, e' \in H^0(w)$. Regardless of whether $e \cap e' = \emptyset$ or not, every $f \in H_Z^1$ intersects at least one of e or e'. Suppose that there is $f \in H_Z$, disjoint from exactly one of the edges e and e', say e. Then f_u, f, $e'w$, and we form a minimal 4-path in H, a contradiction. Since there are exactly two edges of H_Z^1 disjoint from e and two other edges of H_Z^1 disjoint from e', we have $|H_Z^1| \leq 12 - 4 = 8$. In view of Fact 6.8, this implies that

$$|H_Z| = \left|H_Z^1\right| + \left|H_Z^0\right| + \left|H_Z^2\right| \leq 8 + 4 = 12.$$

Similarly, there exists in H_Z^2 at least one edge intersecting e and disjoint from e' and at least one edge intersecting e' and disjoint from e, implying $|H_Z^2| \leq 4 - 2 = 2$. □

Fact 6.10. If $|H^0(u)| \geq 3$ and $|H^0(w)| \geq 3$, then $|H_Z| \leq 13$.

Proof. Observe that, if $e \in H^0(w)$ and $e' \in H^0(u) \cap H^0(w)$ are two disjoint edges, then there is no $f \in H_Z$ with $f \cap e' = \emptyset$, because otherwise edges f, ew, we', and $e'u$ would form a minimal 4-path in H. As there are exactly four triples in $\binom{Z}{3}$ disjoint from e', the presence of such e, e' eliminates 4 edges from H_Z (see Figure 18A,B).

Further note, that since $|H^0(u)| \geq 3$, $|H^0(w)| \geq 3$, and both $H^0(u)$, $H^0(w)$ are subgraphs of the cycle $y_1z_1y_2z_2$, there are at least two edges $e, e' \in H^0(u) \cap H^0(w)$. If $e \cap e' = \emptyset$, then every triple in $\binom{Z}{3}$ disjoint from e' intersects e and vice versa (see Figure 18C). Therefore, $|H_Z| \leq 20 - 2 \cdot 4 = 12$, better than needed. Otherwise e and e' share a vertex, and there are two further edges $\hat{e}, \hat{e}' \in H^0(u) \cup H^0(w)$, such that $e \cap \hat{e} = \emptyset$ and $e' \cap \hat{e}' = \emptyset$ (see Figure 18D). Hence, we can apply the above elimination

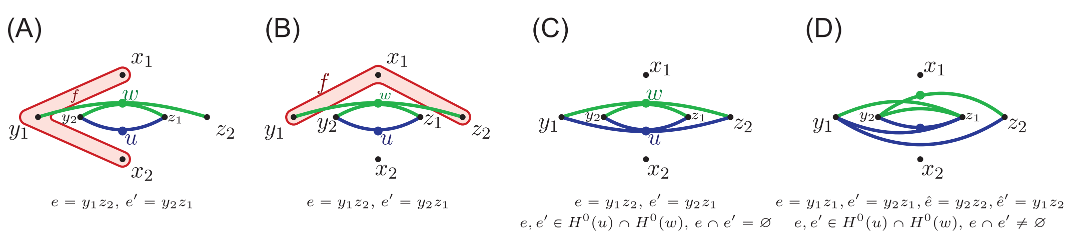


FIGURE 18 (A,B) A minimal 4-path $\{f, ew, we', e'u\}$and (C,D) $e, e' \in H^0(u) \cap H^0(w)$ [Color figure can be viewed at wileyonlinelibrary.com]

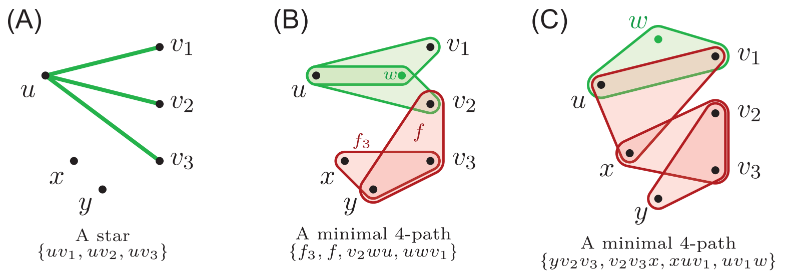


FIGURE 19 (A) A star in $H(w)$and (B,C) minimal 4-paths in H [Color figure can be viewed at wileyonlinelibrary.com]

scheme to these two pairs. As there is exactly one triple in $\binom{Z}{3}$ disjoint from both e and e', by sieve principle, we eliminate from H_Z exactly $4 + 4 - 1 = 7$ edges, leading to the required bound $|H_Z| \leq 20 - 7 = 13$. □

Fact 6.11. If $S_4^{(2)} \subseteq L_H(w)$, then $|H_Z| \leq 14$.

Proof. Recall that $L_H(w) \subseteq \binom{Z}{2}$. Let $\{uv_1, uv_2, uv_3\}$ be in $L_H(w)$ and set $Z \setminus \{u, v_1, v_2, v_3\} = \{x, y\}$ (see Figure 19A). If for some $i \in [3]$, $f_i = xyv_i \in H_Z$, then none of the six triples $f \subset Z$, such that $u \notin f$ and $|f \cap f_i| = 2$, can belong to H_Z, since otherwise the edges f_i, f, uwv_j, uwv_k, $\{i, j, k\} = \{1, 2, 3\}$, would form a minimal 4-path, contradicting the $\mathcal{P}_4$-freeness of H. Thus, $|H_Z| \leq 20 - 6 = 14$ (see Figure 19B). Therefore, assume now that $f_1, f_2, f_3 \notin H_Z$. Again by the $\mathcal{P}_4$-freeness of H, from each of the three disjoint sets of triples,

$$\{uv_1x, v_2v_3x, v_2v_3y\},\ \{uv_2x, v_1v_3x, v_1v_3y\},\ \{uv_3x, v_1v_2x, v_1v_2y\},$$

at most two triples may belong to H_Z and, consequently, $|H_Z| \leq 20 - 3 - 3 = 14$ (see Figure 19C). □

Fact 6.12. If $L_H(w) = S_5^{(2)}$ is a star with the center in $\{y_1, y_2, z_1, z_2\}$ and $|H_Z| \geq 14$, then $H \subseteq SK_n$.

Proof. Without loss of generality we may assume that y_1 is the center of the star $L_H(w)$. Thus, $L_H(w) = \{y_1v : v \in A\}$, where $A = \{x_1, x_2, z_1, z_2\}$. Let us denote by K_A the complete 3-graph on A. We will prove that

$$H \subseteq K_A \cup S(y_1, A) = SK_n,$$

which boils down to showing that for each edge $f \in H$ with $f \not\subseteq A$ we have $f \cap A \neq \varnothing$ and $y_1 \in f$.

Recall that each $f \in H$ satisfies $|f \cap Z| \geq 2$ and $f \cap Z \neq \{y_1, y_2\}$. Therefore, for all $f \in H$, we have $f \cap A \neq \varnothing$. Consequently, we only need to show that if $f \not\subseteq A$, then $y_1 \in f$.

Let us begin with $f \in H_Z$. By Fact 6.4(iii), for all $f \in \left(H_Z^0 \cup H_Z^1\right) \setminus C$ we have $f \cap \{x_1, y_1\} \neq \varnothing$ and $f \cap \{x_2, y_1\} \neq \varnothing$, and thus, $y_1 \in f$. So, we are done with H_Z, except that we still need to rule out the presence of the edge $x_1x_2y_2$ in H.

The above established fact that $y_1 \in f$ for all $f \in \left(H_Z^0 \cup H_Z^1\right) \setminus C$ implies that $|H_Z^0| \leq 3$ and $|H_Z^1| \leq 8$, and, in turn, $|H_Z^2| = |H_Z| - |H_Z^0| - |H_Z^1| \geq 14 - 3 - 8 = 3$. But triples $x_1x_2y_2$, $x_1x_2z_1$, wy_1z_1, and wy_1z_2 form a minimal 4-path, and the same is true with $x_1x_2z_1$ replaced by $x_1x_2z_2$ and the last two edges reversed. Thus, to satisfy $|H_Z^2| \geq 3$, we must have $x_1x_2y_2 \notin H_Z^2$, while $x_1x_2z_1, x_1x_2z_2 \in H_Z^2$.

Turning to the edges of $H \setminus H_Z$, recall that all edges of H containing w contain also y_1. Next, fix an arbitrary vertex $u \in V \setminus (Z \cup \{w\})$, and observe, that $x_1x_2z_1 \in H_Z^2$ entails $ux_1x_2 \notin H$, and thus $H^2(u) = \varnothing$, because otherwise H would contain a minimal 4-path consisting of edges ux_1x_2, $x_1x_2z_1$, wy_1z_1, and wy_1z_2. Finally, note that, by Fact 6.4(ii), all edges of $H^0(u)$ intersect $\{x_1, y_1\}$ and $\{x_2, y_1\}$, while all edges of $H^1(u)$ intersect $\{y_1, z_1\}$ and $\{y_1, z_2\}$. This implies that for all $e \in L_H(u) = H^0(u) \cup H^1(u)$, the condition $y_1 \in e$ holds. In summary, for all $f \in H \setminus H_Z$, we have $y_1 \in f$, which ends the proof. □

6.2 | Proofs of Lemmas 6.1 and 6.2

Proof of Lemma 6.1. Assume, for the sake of a contradiction, that the assumptions of Lemma 6.1 are satisfied, but $H \nsubseteq SP_n$. Let $u, w \in W$ be two vertices with degree in H at least 5. In view of Corollary 6.6, we actually have

$$\deg_H(u) = \deg_H(w) = 5.$$

Then Corollary 6.5 combined with (30) and Fact 6.3 tells us that this is possible only if $|H^2(u)| = |H^2(w)| = 1$ and one of the followings is true:

(i) $|H^0(u)| = |H^0(w)| = 4$;
(ii) $|H^1(u)| = |H^1(w)| = 4$;
(iii) $|H^0(u)| = |H^1(u)| = |H^0(w)| = |H^1(w)| = 2$.

Case (i) is impossible—otherwise, the vertices $y_1z_2uy_2z_1wx_1x_2$ would form a minimal 4-path in H.

If we are in case (ii), then because $H \nsubseteq SP_n$, Fact 6.3 together with Fact 6.4(ix), ensures that both $H^1(u)$ and $H^1(w)$ are 4-cycles, either $C_y = x_1y_1x_2y_2$ or $C_z = x_1z_1x_2z_2$. Now, Fact 6.4(i) entails, that exactly one of them, say $H^1(u)$, equals C_y, whereas the other one $H^1(w) = C_z$. But then $|H^1(u) \cup H^1(w)| \geq 7$, and thus, in view of Fact 6.4(viii), $|H_Z| \leq 8$ yielding

$$|H[Z \cup \{u, w\}]| = |H_Z| + \deg_H(u) + \deg_H(w) \leq 8 + 5 + 5 = 18 < 22.$$

Therefore there exists a vertex $v \in W \setminus \{u, w\}$, with $\deg_H(v) \geq 5$. Another application of Corollary 6.5(ii) with v in place of u says, that $H^0(v) = \varnothing$, again by Facts 6.3 and 6.4(ix), either $H^1(v) = C_y$ or $H^1(v) = C_z$. But, because already $H^1(u) = C_y$ and $H^1(w) = C_z$, in view of Fact 6.4(i) this is impossible, namely, we arrive at a contradiction. Finally, in case (iii), one can observe that, by Fact 6.9, $|H_Z^1| \leq 8$ and $|H_Z^2| \leq 2$, while, by Fact 6.7, $H_Z^0 = \varnothing$ and for every $v \in W \setminus \{u, w\}$, $H^0(v) = \varnothing$. Altogether, we get $|H_Z| \leq 10$ and, consequently, $|H[Z \cup \{u, w\}]| \leq 20$. Hence there is a vertex $v \in W \setminus \{u, w\}$ with $\deg_H(v) \geq 5$. Now, Corollary 6.5(ii) says $|H^1(v)| \leq 2$, and so $\deg_H(v) = |H^0(v)| + |H^1(v)| + |H^2(v)| \leq 3$, yielding a contradiction with $\deg_H(v) \geq 5$. □

Proof of Lemma 6.2. Let $u, w \in W$ be two vertices, such that

(i) $|H[Z \cup \{u, w\}]| \geq 22$ and
(ii) $\deg_H(w) \leq 4$.

We will show that $H \subseteq SK_n$, which will end the proof. Set $\hat{H} = H[Z \cup \{u, w\}]$. Because $L_H(u) \subseteq \binom{Z}{2}$, the connectivity of H implies $H[Z \cup \{u\}] \neq K_6^{(3)} \cup K_1$, and thereby, in view of Lemma 3.1, $|H[Z \cup \{u\}]| \leq 19$. Consequently, $\deg_H(w) \geq 3$, yielding that at least one of the graphs, $H^0(w)$ or $H^1(w)$, is not empty. Hence, by (30), Fact 6.3 and Corollary 6.5, $\deg_H(u) \leq 4 + 2 + 1 = 7$. Similarly, $\deg_H(u) \geq 3$. Suppose that $\deg_H(u) \geq 6$. Then, in view of the bound $\deg_H(w) \geq 3$, Corollary 6.6 tells us that $\deg_H(w) = 3$ and so $H^2(w) \neq \emptyset$. In addition, as $|H^0(u)| + |H^1(u)| \geq 5$, either $|H^0(u)| \geq 2$ or $|H^1(u)| = 4$, implying, together with Facts 6.9 (with u and w swapped) and 6.4(vii), that $|H_Z| \leq 12$. Therefore (i) entails, that $\deg_H(u) = 7$ which, in turn, results $|H^0(u)| \geq 2$ and $H^2(u) \neq \emptyset$. But then Facts 6.9 and 6.7 yield that $|H_Z^1| \leq 8$, $|H_Z^2| \leq 2$, and $H_Z^0 = \emptyset$. Consequently,

$$|\hat{H}| = \left|H_Z^0\right| + \left|H_Z^1\right| + \left|H_Z^2\right| + \deg_H(u) + \deg_H(w) \leq 0 + 8 + 2 + 7 + 3 = 20,$$

contradicting (i).

Hence, from now on, we assume that $\deg_H(u) \leq 5$. Then, in view of (i) and (ii), it follows that $|H_Z| \geq 22 - 5 - 4 = 13$, implying, via Fact 6.4(vii), that both $|H^1(u)| \leq 3$ and $|H^1(w)| \leq 3$. We split the proof into three cases according to the emptiness of $H^2(u)$ and $H^2(w)$. In particular we will show that if at least one of these graphs is not empty, then $|\hat{H}| \leq 21$, contradicting (i).

Case 1. $\mathbf{H^2(u)} \neq \emptyset$ and $\mathbf{H^2(w)} \neq \emptyset$. If, in addition, $H^0(u) = H^0(w) = \emptyset$, then either $|H^1(u) \cup H^1(w)| = 2$ and so, by Fact 6.4(v),

$$|\hat{H}| = |H_Z| + \deg_H(u) + \deg_H(w) \leq 15 + 3 + 3 = 21,$$

or $|H^1(u) \cup H^1(w)| \geq 3$. Then, in view of Fact 6.4(vi), $|H_Z| \leq 13$ and, again, $|\hat{H}| \leq 13 + 4 + 4 = 21$.

Therefore we may assume, that $H^0(u) \cup H^0(w) \neq \emptyset$ yielding, together with Fact 6.7, $H_Z^0 = \emptyset$. Moreover, since $|H_Z| \geq 13$, Fact 6.9 tells us that both $|H^0(u)| \leq 1$ and $|H^0(w)| \leq 1$. Finally, by Fact 6.4(iv)–(vi), either $|H^1(u) \cup H^1(w)| = 1$ and thus $|H_Z^1| \leq 10$, $|H^1(u) \cup H^1(w)| = 2$, entailing $|H_Z^1| \leq 9$, or $|H^1(u) \cup H^1(w)| \geq 3$ and then $|H_Z^1| \leq 8$. That is, $|H_Z^1| + |H^1(u)| + |H^1(w)| \leq 14$ and the equality holds only if $|H^1(u)| = |H^1(w)| = 3$. Altogether, in all of these cases, as $|H_Z^2| \leq 4$, and $|H^0(u)| + |H^2(u)| + |H^0(w)| + |H^2(w)| \leq 4$,

$$|\hat{H}| = \left|H_Z^0\right| + \left|H_Z^2\right| + \left(\left|H_Z^1\right| + |H^1(u)| + |H^1(w)|\right) + |H^0(u)| + |H^2(u)| + |H^0(w)| + |H^2(w)| \leq 21,$$

unless $|H^1(u)| = |H^1(w)| = 3$, in which case $|\hat{H}| \leq 21$ by using $\deg_H(u) + \deg_H(w) \leq 9$ and $|H_Z^1| \leq 8$.

Case 2. $\mathbf{H^2(u)} \neq \emptyset$ and $\mathbf{H^2(w)} = \emptyset$ (the proof of the case $H^2(u) = \emptyset$ and $H^2(w) \neq \emptyset$ is similar). Recall, that $|H^1(w)| \leq 3$ and $|H_Z| \geq 13$ which implies, together with Fact 6.9, that $|H^0(w)| \leq 1$. Therefore, $\deg_H(w) \leq 3$, because otherwise, $|H^1(w)| = 3$ and $|H^0(w)| = 1$. But then, by Facts 6.4(vi) and 6.8, $|H_Z^1| \leq 8$ and $|H_Z^0| + |H_Z^2| \leq 4$, contradicting $|H_Z| \geq 13$. Hence, $\deg_H(u) + \deg_H(w) \leq 8$ from which we infer that $|H_Z| \geq 14$ and, consequently, by Fact 6.4(vi), $|H^1(w)| \leq 2$. Thus, $|H^0(w)| = 1$ and $|H^1(w)| = 2$. But then, again by Facts 6.4(v) and 6.8, $|H_Z| \leq 9 + 4 < 14$, a contradiction.

Case 3. $\mathbf{H^2(u) = H^2(w)} = \varnothing$. First observe, that (30), Fact 6.3, and Corollary 6.5 tell us $\deg_H(u), \deg_H(w) \leq 4$ and, consequently, (i) yields $|H_Z| \geq 14$. Thus, by Fact 6.4(vi),

$$|H^1(u) \cup H^1(w)| \leq 2.$$

Note also that both

$$|H^0(u)| \leq 2 \text{ and } |H^0(w)| \leq 2,$$

because otherwise Corollary 6.5 and $\deg_H(u), \deg_H(w) \geq 3$ entail, that $|H^0(u)| \geq 3$ and $|H^0(w)| \geq 3$. This, however, together with Fact 6.10 implies $|H_Z| \leq 13$, a contradiction.

Now, in view of Fact 6.12, to finish the proof it is enough to show that at least one of the graphs $L_H(u)$ or $L_H(w)$, is a star $S_5^{(2)}$ with the center in $\{y_1, y_2, z_1, z_2\}$. To this end observe, that if $|H^1(u) \cup H^1(w)| = 2$ and either $|H^0(u)| = |H^0(w)| = 1$ or $|H^1(u)| = |H^1(w)| = 1$, then $\deg_H(u) = \deg_H(w) = 3$ and, in view of Fact 6.4(v), $|H_Z| \leq 15$, yielding $|\hat{H}| \leq 15 + 3 + 3 = 21$, a contradiction with (i).

Otherwise there exists an edge $e \in H^1(u) \cap H^1(w)$, and at least one of the graphs, $H^0(u)$ or $H^0(w)$, say $H^0(w)$, has two edges. We let $\{v\} = e \cap \{y_1, y_2, z_1, z_2\}$ and note, that due to Fact 6.4(ii), every edge of $H^0(u) \cup H^0(w)$ contains v. Therefore $S_4^{(2)} \subseteq L_H(w)$ entailing, together with Fact 6.11, $|H_Z| \leq 14$, and thereby $\deg_H(u) = \deg_H(w) = 4$. In particular, $H^0(w)$ has two edges both containing v. Finally, a repeated application of Fact 6.4(ii) reviles, that $L_H(u)$ is a star $S_5^{(2)}$ with the center v, as required. □

7 | RAMSEY NUMBERS

7.1 | Shorter paths

Before we turn to proving Theorem 1.2, let us briefly discuss Ramsey numbers for 3-uniform minimal paths of shorter length. Observe that the family $\mathcal{P}_2$ consists of two 3-graphs, each being a pair of overlapping edges, either in one (a bow) or two vertices (a kite). Therefore, $\mathcal{P}_2$-free 3-graphs are necessarily matchings, that is, consist of disjoint edges only. Consequently, $\mathrm{ex}_3(n; \mathcal{P}_2) = \lfloor n/3 \rfloor$, and, by (1),

$$R(\mathcal{P}_2; r) = \min\left\{n: \frac{\binom{n}{3}}{\lfloor n/3 \rfloor} > r\right\},$$

or, asymptotically, $R(\mathcal{P}_2; r) \sim \sqrt{2r}$, as $r \to \infty$. For small r, in particular, $R(\mathcal{P}_2; 2) = R(\mathcal{P}_2; 3) = 4$, while $R(\mathcal{P}_2; 4) = 5$. In [1], the two 3-graphs belonging to $\mathcal{P}_2$ were considered separately. It was shown there that $R(bow; r) \sim \sqrt{6r}$, while $R(kite; r) \in \{r + 1, r + 2, r + 3\}$ depending on the divisibility of r by 6. It is, perhaps, interesting to see the drop from $\sqrt{6r}$ to $\sqrt{2r}$ when the bow is accompanied by the kite.

The family $\mathcal{P}_3$ also consists of two 3-graphs, among them the linear path P_3. For the latter, an easy lower bound by a construction of Gyárfás and Raeisi [10] says that $R(P_3; r) \geq r + 6$. It was proved in a series of papers [13,15,21,20] that, indeed, $R(P_3; r) = r + 6$ for $r \leq 10$. The trivial upper bound, $R(P_3; r) \leq 3r$, stemming from (1) was improved down to $R(P_3; r) < 1.98r$ in [18].

Turning to minimal paths of length 3, there is a similar lower bound $R(\mathcal{P}_3; r) \geq r + 5$. Using the known value of $\mathrm{ex}_3(7; \mathcal{P}_3) = 15$ determined in [19], it follows by (1) that indeed $R(\mathcal{P}_3; 2) = 7$. With a bit more effort, observing that a connected $\mathcal{P}_3$-free 3-graph must be intersecting and using the Hilton–Milner Theorem 1.3, one can also show that $R(\mathcal{P}_3; r) = r + 5$ for $r \leq 7$. The range of r, for which $R(\mathcal{P}_3; r) = r + 5$ is certainly wider, but to prove it one would need more sophisticated tools, like the third-order Turán number $\mathrm{ex}_3^{(3)}(n; \mathcal{P}_3)$.

7.2 | Proof of Theorem 1.2

Let us start with a general lower bound on $R(\mathcal{P}_4; r)$ based on the slightly modified construction given by Gyárfás and Raeisi in [10]. We let

$$s_r = \max\left\{s \in \mathbb{Z} : \sum_{k=6}^{s} \binom{k}{2} \leq r - 1\right\} \text{ and } t_r = \max\left\{t \in \mathbb{Z} : \binom{t}{3} \leq r\right\}.$$

Proposition 7.1. *For all $r \geq 1$,*

$$R(\mathcal{P}_4; r) \geq r + \max\{s_r, t_r\} + 1 \geq r + \sqrt[3]{6r} + 1.$$

Note, that

$$s_r = \begin{cases} 5 & \text{for} \quad 1 \leq r \leq 15, \\ 6 & \text{for} \quad 16 \leq r \leq 36, \\ 7 & \text{for} \quad 37 \leq r \leq 64, \\ 8 & \text{for} \quad 65 \leq r \leq 100, \\ 9 & \text{for} \quad 101 \leq r \leq 145, \\ \cdots \end{cases} \quad t_r = \begin{cases} 3 & \text{for} \quad 1 \leq r \leq 3, \\ 4 & \text{for} \quad 4 \leq r \leq 9, \\ 5 & \text{for} \quad 10 \leq r \leq 19, \\ 6 & \text{for} \quad 20 \leq r \leq 34, \\ 7 & \text{for} \quad 35 \leq r \leq 55, \\ \cdots \end{cases} \text{and thus } R(\mathcal{P}_4; r)$$

$$\geq \begin{cases} r + 6 & \text{for} \quad r \geq 1, \\ r + 7 & \text{for} \quad r \geq 16, \\ r + 8 & \text{for} \quad r \geq 35, \\ r + 9 & \text{for} \quad r \geq 56, \\ r + 10 & \text{for} \quad r \geq 84, \\ \cdots \end{cases}$$

In particular, for $r \geq 20$ we have $t_r \geq s_r$.

Proof. Set $m = \max\{s_r, t_r\}$ and let $V\left(K_{r+m}^{(3)}\right) = \{1, 2, \ldots, r + m\}$. If $m = s_r$, for $i = 1, \ldots, r - 1$, color every edge of $K_{r+m}^{(3)}$ whose minimum vertex is i by color i. In addition, apply different colors from $\{1, \ldots, r - 1\}$ to all edges with minimum vertex in the set $\{r, r + 1, \ldots, r + m - 6\}$. Note that there are exactly $\sum_{k=6}^{m} \binom{k}{2} \leq r - 1$ such edges. Moreover, the edges of color i form a starplus, so no monochromatic copy of a minimal 4-path has been created in any of the first $r - 1$ colors. The remaining uncolored edges form a complete 3-graph $K_6^{(3)}$ on the last 6 vertices $r + m - 5, \ldots, r + m$ and we color them by color r. As a minimal 4-path has at least 7 vertices, there is no member of $\mathcal{P}_4$ in color r as well.

If $m = t_r$, the construction is even simpler. For $i = 1, \ldots, r$, color every edge of K_{r+m} whose minimum vertex is i by color i. In addition, apply different colors from $\{1, \ldots, r\}$ to all $\binom{m}{3} \le r$ edges spanned on the vertices $r + 1, \ldots, r + m$. Again, each color is a starplus, so no monochromatic copy of a minimal 4-path has been created. □

Proof of Theorem 1.2. In view of Proposition 7.1, we only need to show the upper bound on $R(\mathcal{P}_4; r)$. For $r = 1$ there is nothing to prove so let us begin with $r = 2$ and $n = 8$. For this purpose observe that in every 2-coloring of $K_8^{(3)}$ at least one color takes at least $\binom{8}{3}/2 = 28 > 22 = \mathrm{ex}(8; \mathcal{P}_4)$ edges, and so, due to Theorem 1.1, contains a member of $\mathcal{P}_4$. Moreover, the same averaging argument entails that this is true for every 8-vertex 3-graph with at least 45 edges.

Now, let $r = 3$ and $n = 9$. With an eye on the case $r = 4$, we are going to prove, for $r = 3$, a slightly stronger result. An r-coloring which does not yield a monochromatic member of $\mathcal{P}_4$ is referred to as *proper*. Let H_9 be a 9-vertex 3-graph with at least $\binom{9}{3} - 2 = 82$ edges and let a proper 3-coloring of H_9 be given. Then, there is a color with at least $\lceil 82/3 \rceil = 28 > 27 = \mathrm{ex}^{(2)}(9; \mathcal{P}_4)$ edges and thus, since the coloring is proper, by Theorems 1.1 and 1.4, that color must be a subset of S_9^{+1}. After removing the center of that star as well as the unique edge not containing it, we obtain a proper 2-coloring of an 8-vertex 3-graph with at least $\binom{8}{3} - 3 = 53$ edges, which, as it is shown above, contains a monochromatic member of $\mathcal{P}_4$, a contradiction.

Finally, consider the case $r = 4$ and $n = 10$. To this end let a proper 4-coloring of all $\binom{10}{3} = 120$ edges of $K_{10}^{(3)}$ be given. If there is a color which is a subset of either S_{10}^{+1} or SP_{10}, then we remove its center together with at most two additional edges. As a result, we obtain a proper 3-coloring of a 9-vertex 3-graph with at least $\binom{9}{3} - 2 = 82$ edges, which, as shown above, contains a monochromatic copy of a member of $\mathcal{P}_4$, a contradiction. Otherwise, in view of Theorems 1.1, 1.4, and 1.5, each of the four colors has exactly 30 edges and is isomorphic to SK_{10}. But this is impossible, because in $K_{10}^{(3)}$ every vertex has degree $\binom{9}{2} = 36$, whereas in SK_{10} each vertex has its degree in $\{4, 11, 26\}$. Clearly, 36 can not be obtained as a sum of four numbers from $\{4, 11, 26\}$ and we are done. □

8 | OPEN PROBLEMS

It would be interesting, though tedious, to calculate higher-order Turán numbers for $\mathcal{P}_4$, that is, $\mathrm{ex}_3^{(s)}(n; \mathcal{P}_4)$, $s \ge 4$, and, using them, to pin down Ramsey numbers $R(\mathcal{P}_4, r)$ for $5 \le r \le r_0$, for some $r_0 \ge 5$.

Another challenging project would be to determine *for all n* the Turán number $\mathrm{ex}_3(n; \mathcal{C}_4)$, where, recall $\mathcal{C}_4$ is the family of all minimal 3-uniform cycles with four edges. Kostochka, Mubayi, and Verstraete showed in [17] that for *large n*

$$\mathrm{ex}_3\left(n; \mathcal{C}_4^3\right) = \binom{n-1}{2} + \left\lfloor \frac{n-1}{3} \right\rfloor.$$

Gunderson, Polcyn, and Ruciński in [9] confirmed this formula for $n \le 7$.

Turán numbers for longer minimal paths and cycles seem to be currently out of reach if one desires the exact values *for all n*.

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