The Origins of the Theory of Random Graphs

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1. Introduction

The origins of the theory of random graphs are easy to pin down. Undoubtedly one should look at a sequence of eight papers co-authored by two great mathematicians: Paul Erdős and Alfred Rényi, published between 1959 and 1968:

[ER59] *On random graphs I*, Publ. Math. Debrecen **6** (1959), 290–297.
[ER60] *On the evolution of random graphs*, Publ. Math. Inst. Hung. Acad. Sci. **5** (1960), 17–61.
[ER61a] *On the evolution of random graphs*, Bull. Inst. Internat. Statist. **38**, 343–347.
[ER61b] *On the strength of connectedness of a random graph*, Acta Math. Acad. Sci. Hungar. **12** (1961), 261–267.
[ER63] *Asymmetric graphs*, Acta Math. Acad. Sci. Hung. **14**, 295–315.
[ER64] *On random matrices*, Publ. Math. Inst. Hung. Acad. Sci. **8** (1964), 455–461.
[ER66] *On the existence of a factor of degree one of a connected random graph*, Acta Math. Acad. Sci. Hung. **17** (1986), 359–368.
[ER68] *On random matrices II*, Studia Sci. Math. Hung. **3** (1968), 459–464.

Our main goal is to summarize the results, ideas and open problems contained in those contributions and to show how they influenced future research in random graphs.

For us it was a great adventure to return to the roots of the theory of random graphs, and to find out again and again, how far-reaching the impact of Erdős and Rényi's work on the field is. The reader will find in our paper many quotations from their original papers (*always in italics*). We use this convention to let them speak directly and to preserve their special insightful

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style and way of thinking and stating the problems. Starting from there we lead the reader through the literature, including the most current one, trying to show how the ideas of Erdős and Rényi developed, how much time, skills and effort to solve some of their most challenging open problems was needed. Finally, to add some "salt and pepper" to our presentation, full of admiration and respect, we point out to a few false statements and oversimplifications of proofs, which have been found in their monumental legacy by the next generations of random graph theorists.

2. The First Question: Connectivity

Although the notion of a random graph appeared in connection to the probabilistic method already in the Erdős paper [25] (see J. Spencer's article in this volume), it was forgotten for a decade until Paul Erdős and Alfred Rényi published a series of papers entirely devoted to properties of random graphs. The model of a random graph they exclusively investigated was the uniform one. Here is how they defined it: "*Let $E_{n,N}$ denote the set of all graphs having n given labeled vertices and N edges. A random graph $\Gamma_{n,N}$ can be defined as an element of $E_{n,N}$ chosen at random, so that each of the elements of $E_{n,N}$ have the same probability to be chosen, namely $1/\binom{\binom{n}{2}}{N}$.*" (In this paper we adopt the original notation $\Gamma_{n,N}$.)

They were aware of existing results about other models of random graphs. In particular, they acknowledge in a footnote to [ER61a] that E. N. Gilbert [36] studied the connectedness of what we call today the binomial model, where "*We may decide with respect to each of the $\binom{n}{2}$ edges, whether they should form part of the random graph considered or not, the probability of including a given edge being $p = N/\binom{n}{2}$ for each edge and the decisions concerning different edges being independent.*" (In this paper we shall denote this model by $\Gamma_{n,p}$.) In [ER61a] they mention that the investigations of the binomial model can be reduced, due to a conditional argument they attribute to Hajek, to that of $\Gamma_{n,N}$. However, they did not formulate any equivalence theorem (these appeared much later in [14] and [59]) and occasionally stated the binomial counterparts of their theorems without proofs or repeated their proofs step by step.

Apparently they were not aware of the result of Gilbert and of the binomial model at all when they wrote their first paper on random graphs,"On random graphs I". The question addressed there was that of connectedness of a random graph. In fact, according to a remark in [ER59], this problem was tried and partially solved already in 1939, when P. Erdős and H. Whitney, in an unpublished work: "*proved that if $N > \left(\frac{1}{2}+\varepsilon\right) n \log n$ where $\varepsilon > 0$ then the probability of $\Gamma_{n,N}$ being connected tends to 1 if $n \to \infty$, but if $N < \left(\frac{1}{2}-\varepsilon\right) n \log n$ with $\varepsilon > 0$ then the probability of $\Gamma_{n,N}$ being connected, tends to 0 if $n \to \infty$.*"

In the first "official" paper on random graphs, Erdős and Rényi refined the above result as their (partial) answer to questions 1–3 from the following list of problems they posed.

1. *What is the probability of* $\Gamma_{n,N}$ *being completely connected?*
2. *What is the probability that the greatest connected component (subgraph) of* $\Gamma_{n,N}$ *should have effectively* $n-k$ *points?* ($k = 0, 1, \ldots$)
3. *What is the probability that* $\Gamma_{n,N}$ *should consist of exactly* $k+1$ *connected components?* ($k = 0, 1, \ldots$)
4. *If the edges of a graph with* n *vertices are chosen successively so that after each step every edge which has not yet been chosen has the same probability to be chosen as the next, and if we continue this process until the graph becomes completely connected, what is the probability that the number of necessary steps* ν *will be equal to a given number* l*?*

Note that in problem 4 Erdős and Rényi describe a genuine random graph process, whose advanced analysis could be carried over only two decades later.

Before turning to the proofs, they recall a recursive formula and a generating function for the number $C(n, N)$ of connected graphs on n labeled vertices and with N edges, due to Riddell and Uhlenbeck, and also Gilbert. But immediately they comment that neither of them "*...helps much to deduce the asymptotic properties of* $C(n, N)$*. In the present paper we follow a more direct approach.*"

We now present the first result on random graphs and its proof in a slightly modified form. The idea of the proof, however, remains unchanged. In the 1959 paper only the middle part of the theorem below was stated explicitly. The other two follow by letting $c = c_n$ tend to $+\infty$ or $-\infty$, respectively.

Theorem 2.1 ([**26**]).

$$P(\Gamma_{n,N} \textit{ is connected }) \to \begin{cases} 0 & \textit{if } \frac{N}{n} - \frac{1}{2}\log n \to -\infty \\ e^{-e^{-2c}} & \textit{if } \frac{N}{n} - \frac{1}{2}\log n \to c \\ 1 & \textit{if } \frac{N}{n} - \frac{1}{2}\log n \to \infty. \end{cases}$$

Proof. For convenience we switch to the binomial model, shortening the original argument a lot, and, at the same time, avoiding a harmless error in the proof of "*the rather surprising Lemma*" of [ER59], pointed out by Godehardt and Steinbach [37].

To make this argument formal, assume that $2np - \log n - \log\log n \to \infty$ but $np = O(\log n)$. Thus, *almost surely* (i.e., with probability tending to 1 as $n \to \infty$), there are no isolated edges in $\Gamma_{n,p}$. What remains to be shown is that there are no components of size $3 \le k \le \frac{n}{2}$ either. To this end consider the random variable X counting such components. Then, bounding the probability that a given set of k vertices spans a connected subgraph by $k^{k-2}p^{k-1}$, and using the inequality $np > \frac{1}{2}\log n$, we obtain

$$\begin{aligned} Exp(X) &\le \sum_{k=3}^{n/2} \binom{n}{k} k^{k-2}p^{k-1}(1-p)^{k(n-k)} < \sum_{k} \left(\frac{en}{k}\right)^k k^{k-2}p^{k-1}e^{-(n-k)pk} \\ &\le \frac{1}{p}\sum_{k=3}^{\sqrt{n}} \frac{1}{k^2}\left(\frac{enp}{e^{(n-\sqrt{n})p}}\right)^k + \frac{1}{p}\sum_{k\ge\sqrt{n}}^{n} \frac{1}{n}\left(\frac{enp}{e^{np/2}}\right)^k \\ &= O\left(\frac{n}{\log n}\frac{\log^3 n}{n^{3/2}}\right) + \frac{1}{\log n}\left(\frac{e\log n}{2n^{1/4}}\right)^{\sqrt{n}} = o(1). \end{aligned}$$

Hence, almost surely there are no components outside the largest one other than isolated vertices (Erdős and Rényi say that such a graph is of type A) and the threshold for connectedness coincides with that for disappearance of isolated vertices, i.e., for $2np - \log n - \log\log n \to \infty$

$$P(\Gamma_{n,p} \text{ is connected }) = P(\delta(\Gamma_{n,p}) > 0) + o(1).$$

Erdős and Rényi found the limiting value of $P(\delta(\Gamma_{n,p}) > 0)$ by inclusion-exclusion. Nowadays a standard approach is by the method of moments which serves to show that the number of isolates is asymptotically Poisson. They used that method in the 1960 paper in a more general setting where components isomorphic to a given graph G were considered. We shall return to this later.

Answering question 4, they gave a somewhat oversimplified proof of the fact that

$$\lim_{n\to\infty} P\left(\frac{\nu - \frac{1}{2}n\log n}{n} < x\right) = e^{-e^{-2x}}. \qquad \square$$

Erdős and Rényi conclude the 1959 paper as follows. "*The following more general question can be asked: Consider the random graph $\Gamma_{n,N(n)}$ with n possible vertices and $N(n)$ edges. What is the distribution of the number of vertices of the greatest connected component of $\Gamma_{n,N(n)}$ and the distribution of the number of its components? What is the typical structure of $\Gamma_{n,N(n)}$ (in the sense in which, according to our Lemma, the typical structure of $\Gamma_{n,N(n)}$ is that it belongs to type A)? We have solved these problems in the present paper only in the case $N(n) = \frac{1}{2}n\log n + cn$. We shall return to the general case in another paper [8].*" ([8] = [ER60] on our reference list.)

As far as connectedness is concerned, in the 1961 paper Erdős and Rényi go on and find the threshold for r-connectivity of $\Gamma_{n,p}$ for every natural r. "*If G is an arbitrary non-complete graph, let $c_p(G)$ denote the least number k such that by deleting k appropriately chosen vertices from G (...) the resulting graph is not connected. (...) Let $c_e(G)$ denote the* *least number l such that by deleting l appropriately chosen edges from G the resulting graph is not connected.*" A graph is *r-connected* if no removal of r or less vertices can disconnect it. When the random graph becomes almost surely r-connected? Theorem 2.1 revealed an interesting feature of random graphs. Namely, quite often trivial necessary conditions become asymptotically sufficient in the sense that for a typical, large graph their fulfillment guaranties that the property in question holds. Due to Theorem 2.1 this is the case of connectedness versus the nonexistence of isolated vertices. For r-connectedness such natural necessary condition is that the minimum degree (denoted in [ER61b] by $c(G)$) must be at least r. Otherwise removing the vertices adjacent to a vertex of minimum degree would disconnect the graph. Erdős and Rényi showed in 1961 that in the range $\frac{1}{2}n\log n \le N \le n\log n$ this is the only way one can disconnect the random graph $\Gamma_{n,N}$ by removing the smallest possible number of vertices. A *minimal cutset* is a set of vertices whose removal makes the graph disconnected but no proper subset of that set has this property. For $2 \le k \le \frac{n-1}{2}$ let $\mathcal{A}_k$ be the event that there is in $\Gamma_{n,N}$ a minimal cutset of size s, $1 \le s \le r-1$, which leaves the second largest component of size k. Arguing similarly as in the proof of Theorem 2.1, they proved that $P(\bigcup_{k\ge 2}\mathcal{A}_k) = o(1)$, meaning that, almost surely, if $\Gamma_{n,N}$ is not r-connected then the only reason for that is the presence of vertices of degree less than r. The method of moments (again, in the inclusion-exclusion cover-up) gives that, for $N(n) = \frac{1}{2}n\log n + \frac{r}{2}n\log\log n + an + o(n)$, their number is asymptotically Poisson. We thus arrived at the main result of the 1961 paper.

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Theorem 2.2 ([**28**]). *If we have*

$$N(n) = \frac{1}{2}n\log n + \frac{r}{2}n\log\log n + an + o(n)$$

where a is a real constant and r a non-negative integer, then

$$\lim_{n\to\infty} P(c_p(\Gamma_{n,N(n)}) = r) = 1 - \exp\left(-\frac{e^{-2a}}{r!}\right), \qquad (3)$$

further

$$\lim_{n\to\infty} P(c_e(\Gamma_{n,N(n)}) = r) = 1 - \exp\left(-\frac{e^{-2a}}{r!}\right) \qquad (4)$$

and

$$\lim_{n\to\infty} P(c(\Gamma_{n,N(n)}) = r) = 1 - \exp\left(-\frac{e^{-2a}}{r!}\right). \qquad (5)$$

In a proceeding remark they promise: "*The statement (5) of Theorem 2.2 gives information about the minimal valency of points of $\Gamma_{n,N}$. In a forth-*

coming note we shall deal with the same question for larger ranges of N *(when* $c(\Gamma_{n,N})$ *tends to infinity with* n*), further with the related question about maximal valency of points of* $\Gamma_{n,N}$*.*" This promise was never fulfilled. The only trace of their interest in the vertex degrees of a random graph can be found in the description of the last phase of the evolution of $\Gamma_{n,N}$ in [ER61a]: "*Phase 5. consists of the range* $N(n) \sim (n \log n)w(n)$ *where* $w(n) \to \infty$*. In this range the whole graph is not only almost surely connected, but the orders of points are almost surely asymptotically equal. Thus the graph becomes in this phase 'asymptotically regular'.*" The proof of that statement can be found in the last section of [ER60]. A very careful analysis of vertex degrees in a random graph is due to Bollobás [10, 11] and can be found also in his book [14].

3. Subgraphs: The Beginning of a Theory

After having written their paper on connectivity of a random graph Erdős and Rényi decide to write a long paper addressing several properties of random graphs. That seminal paper was preceded by an extended abstract [ER61a], where they outlined the main goals of the theory to be born. "*Our main goal is to show (...) that the evolution of a random graph shows very clear-cut features. The theorems we have proved belong to two classes. The theorems of the first class deal with the appearance of certain subgraphs (e.g., tress, cycles of a given order etc.) or components, or other local structural properties, and show that for many types of local structural properties* A *a definite 'threshold'* $A(n)$ *can be given, so that if* $\frac{N(n)}{A(n)} \to 0$ *for* $n \to \infty$ *then the probability that the random graph* $\Gamma_{n,N(n)}$ *has the structural property* A *tends to* 0 *for* $n \to \infty$*, while for* $\frac{N(n)}{A(n)} \to \infty$ *for* $n \to \infty$ *the probability that the random graph* $\Gamma_{n,N(n)}$ *has the structural property* A *tends to* 1 *for* $n \to \infty$*. (...) The theorems of the second class are of similar type, only the properties* A *considered are not of a local character, but global properties of the graph* $\Gamma_{n,N(n)}$ *(e.g., connectivity, total number of components, etc.).*" The existence of a threshold in all cases they considered was a rather surprising fact for Erdős and Rényi. Only three decades later it was proved by Bollobás and Thomason [19] that, as a consequence of the Kruskal-Katona inequality, every monotone property (family) of random subsets of a set has a threshold in the above sense.

In the same abstract they comment that their proofs are "*... completely elementary, and are based on the asymptotic evaluation of combinatorial formulae and on some well-known general methods of probability theory ....*"

The first theorem of the major paper [ER60] established the threshold for the existence of a subgraph of a given type for a broad class of subgraphs. "*If a graph has* n *vertices and* N *edges, we call the number* $\frac{2N}{n}$ *the 'degree' of the graph (as a matter of fact* $\frac{2N}{n}$ *is the average degree of the vertices of* G*.) If a graph* G *has the property that* G *has no subgraph having a larger degree than* G *itself, we call* G *a* balanced *graph.*"

Theorem 3.1 ([**27**]). *Let $k \geq 2$ and l $(k-1 \leq l \leq \binom{k}{2})$ be positive integers. Let $\mathcal{B}_{k,l}$ denote an arbitrary not empty class of connected balanced graphs consisting of k points and l edges. The threshold function for the property that the random graph considered should contain at least one subgraph isomorphic with some element of $\mathcal{B}_{k,l}$ is $n^{2-\frac{k}{l}}$.*

Among special cases they mention trees, connected unicyclic graphs, cycles, complete graphs and complete bipartite graphs all of which are balanced. Over 20 years later, Bollobás [9] generalized this theorem to arbitrary (not only balanced) graphs. He, however, used a rather complicated method. In 1985, to a great surprise to all involved, Ruciński and Vince [73] found out that the original proof of Erdős and Rényi which was based on the second moment method can be easily adapted to cover all graphs as well. We now state that result in the binomial model.

Theorem 3.2 ([**9**]). *For an arbitrary graph G with at least one edge,*

$$\lim_{n\to\infty} P(G \subset \Gamma_{n,p}) = \begin{cases} 0 & \text{if } p = o(n^{-1/m_G}) \\ 1 & \text{if } n^{-1/m_G} = o(p), \end{cases}$$

where $m_G = \max_{H\subseteq G} d_H$ and $d_G = \frac{|E(G)|}{|V(G)|}$.

A crucial role in the Ruciński-Vince proof of Theorem 3.2 is played by the quantity $\Phi_G = \min_{H\subseteq G} Exp(X_H)$. In fact, the inequalities

$$1 - \Phi_G \leq P(G \not\subset \Gamma_{n,p}) \leq c_1/\Phi_G$$

obtained in that proof have been strengthened to exponential bounds

$$e^{-c_2\Phi_G} \leq P(G \not\subset \Gamma_{n,p}) \leq e^{-c_3\Phi_G},$$

where the L-H-R follows by the FKG inequality and the R-H-S is a special case of a recent inequality from [42].

As far as the asymptotic distributions of subgraph counts are concerned, Erdős and Rényi treated in [ER60] only trees and cycles. For trees of order k they established a limiting Poisson distribution on the threshold $N \sim cn^{\frac{k-2}{k-1}}$. They observed that the same result holds for isolated trees, since in this range almost surely all k-vertex trees are isolated (i.e., are components of the random graph). They also found another Poisson threshold for isolated trees at $N = \frac{1}{2k} n\log n + \frac{k-1}{2k} n \log\log n + cn + o(n)$, beyond which isolated trees die out (swallowed by the giant component on its way to absorb all the vertices of the random graph). They also established an asymptotic normality of the number of isolated trees of order k (after suitable standardization) in the whole range of N between the two thresholds. As observed by A. Barbour in [5], the proof given by Erdős and Rényi was not correct and in the range $N \sim cn$, $c \neq 1/2$, the standardization was not right. However, using another method Barbour showed that indeed the asymptotic normality holds in the entire range in question. For cycles and isolated cycles they established a Poisson distribution (different in each case) at $N \sim cn$ and observed that contrary to isolated trees, "*... the probability that $\Gamma_{n,N}$ contains an isolated cycle of order k never approaches* 1." A similar result was proved for connected unicyclic graphs. All these results were obtained by the method of moments based on a fact from probability theory that for all distributions which are uniquely determined by their moments (Poisson and normal are such) the convergence of all moments of a sequence of random variables to the moments of that distribution implies convergence in distribution [8, Theorem 30.2]. Erdős and Rényi prove this fact as a lemma just for the Poisson distribution, although they use it also for the normal distribution. At the end of the paper, in a remark added in proof, they acknowledge that N. V. Smirnov proved this lemma already in 1939.

They conclude their investigations of local properties of random graphs with the comment: "*Similar results can be proved for other types of subgraphs, e.g., complete subgraphs of a given order. As however these results and their proofs have the same pattern as those given above we do not dwell on the subject any longer and pass to investigate global properties of the random graph $\Gamma_{n,N}$.*" In 1979, K. Schürger, a former Ph.D. student of Erdős, proved similar results for complete subgraphs [74] and a few years later Karoński [47] extended them to so called k-trees, a common generalization of trees and complete graphs. All these particular cases led to a general result for all strictly balanced graphs. A graph is *strictly balanced* if every proper subgraph has its degree strictly smaller than the graph itself. Let us denote $d_G = \frac{|E(G)|}{|V(G)|}$ and recall that X_G is the number of copies of G in a random graph $\Gamma_{n,p}$. The following result was proved independently in [9] and [48].

Theorem 3.3 (**[9, 48]**). *If G is a strictly balanced graph and $np^{d_G} \to c > 0$ then X_G converges to the Poisson distribution with expectation $\frac{c^v}{aut(G)}$.*

If a graph G is balanced but not strictly balanced then the limiting distribution of X_G on the threshold, i.e. when $p = \Theta(n^{-1/d_G})$, becomes quite involved. Although, in principle, as shown by Bollobás and Wierman [20], it can be computed, there is no nice closed formula. For example, when G is a disjoint union of 2 triangles then the limit distribution is that of the random variable $\binom{Y}{2}$, where Y is Poisson. When G is the triangle with a pendant edge, the limit is $\sum_{i=1}^{Z} Y_i$, where all random variables involved are independent and Poisson. When G is the triangle with two pendant edges hanging at the same vertex then X_G converges to the distribution of $\sum_{i=1}^{Z} \binom{Y_i}{2}$, where again all random variables are independent Poisson. One more example: if G is the triangle with a path of length 2 hanging at one of it vertices, then the limit distribution is that of $\sum_{i=1}^{\sum_{j=1}^{U} W_j} Y_i$, where all random variables are independent Poisson. We can only hope that so far the reader is convinced that a pattern does indeed exist.

If G is nonbalanced, then the expectation of X_G tends to infinity and one has to normalize. It turns out that there is a nonrandom sequence $a_n(G) \to \infty$ such that the asymptotic distribution of $\frac{X_G}{a_n(G)}$ coincides with that of X_H, where H is the largest subgraph of G for which $d_H = m_G$. Clearly, H is balanced and we are back to the balanced case. The sequence $a_n(G)$ is equal to the expected number of extensions of a given copy of H to a copy of G in the random graph $\Gamma_{n,p}$. For details see [71, page 292].

Beyond the threshold, i.e., when $np^{m_G} \to \infty$, X_G converges after standardization to the standard normal distribution as long as $n^2(1-p) \to \infty$. (For bigger p X_G is either Poisson or degenerate, according to the formula $X_G \sim \binom{n}{v}\frac{v!}{aut(G)} - c_n(G)Z$, where Z is the binomial random variable counting edges in the complement of $\Gamma_{n,p}$ and $c_n(G)$ is the number of copies of G in K_n containing a fixed edge. For details see [70].) This result was supplemented by the rate of convergence in [7]. It was shown there that the total variation distance between standardized X_G and the standard normal distribution can be bounded by $O(\frac{1}{\sqrt{\Phi_G}})$ as long as $p \not\to 1$ and by $O(\frac{1}{n\sqrt{1-p}})$ otherwise. Recall that $\Phi_G \to \infty$ if and only if $np^{m_G} \to \infty$.

A variant of the small subgraph problem is one when we only count induced subgraphs of $\Gamma_{n,p}$ which are isomorphic to G (*induced copies*). Let Y_G count them. Then, denoting $v = |V(G)|$ and $l = |E(G)|$, $Exp(Y_G) = Exp(X_G)(1-p)^{\binom{v}{2}-l}$, and as long as $p \to 0$ there is no substantial difference in the limiting distribution of X_G and Y_G. For p constant, however, interesting things may happen. First of all, in contrast to X_G, the variance of Y_G may drop below the order of n^{2v-2}. It does so when $Exp(I|J_{12}) = Exp(I)$, i.e., when $p = l/\binom{v}{2}$, where I is the indicator of the event that there is an induced copy of G in $\Gamma_{n,p}$ on the vertex set $\{1,\ldots,v\}$ and J_{ij} is the indicator that the edge ij is present in $\Gamma_{n,p}$. But if $Var(Y_G) = \Theta(n^{2v-3})$ then still Y_G is asymptotically normal, and only when the variance drops further down to the order of n^{2v-4} the distribution of standardized Y_G becomes nonnormal (the convolution of normal and χ^2 distributions). It is a purely combinatorial question when $Var(Y_G) = \Theta(n^{2v-4})$. For the higher terms to cancel out one needs that $Exp(I|J_{12}, J_{13}, J_{23}) = Exp(I)$, or, equivalently, that in addition to $p = l/\binom{v}{2}$, the proportion $t_3 : t_2 : t_1 : t_0 = p^3 : 3p^2q : 3pq^2 : q^3$ is satisfied, where t_i is the number of induced subgraphs of G isomorphic to the graph with 3 vertices and i edges. For $p = \frac{1}{2}$, an example of a graph satisfying these requirements is the wheel on 8 vertices, i.e. the graph obtained from the 7-cycle by joining a new vertex to every vertex of the cycle. For some time it was an open question if such abnormal cases take place for every rational p. A positive answer to that puzzle is due to combined efforts of
AQ3 Janson, Kratochvíl, Kärrman and Spencer [41, 45, 49].

The random variables X_G and Y_G are examples of sums of random variables with only few dependent summands. In particular, the summands forming Y_G are dependent only if the sets corresponding to the indices intersect (on at least 2 vertices, in fact). The reason is that the property of the vertex set we are after depends only on the presence and absence of the edges within the set. The situation changes when we move to the properties depending also on the pairs with one endpoint in the set. Then all summands are mutually dependent, but most just weakly. We have already encountered such a case when studying the number of components of $\Gamma_{n,p}$ which are isomorphic to a given graph G. Clearly this property requires that there is no edge with one endpoint in the set of vertices of a copy of G. Another example of such "semi-induced" property is the notion of *a maximal clique*. This is a complete subgraph not contained in any bigger complete subgraph of a graph. For a vertex set to span a maximal clique one needs that no other vertex is adjacent to all the vertices of the set. In [6] the limiting distribution of the number of maximal k-cliques was investigated. It was proved that for $k \geq 2$ there are two Poisson thresholds for the existence of maximal k-cliques and the phase of asymptotic normality between them. Finally, there are characteristics which lead to sums of random variables indexed by vertex sets, which each depend on the presence or absence of all the edges in $\Gamma_{n,p}$. An example of this is the number of copies of G disjoint from all other copies of G in $\Gamma_{n,p}$. Here even the expectation is difficult to obtain, and the limiting normal distribution is still beyond ones reach.

4. Phase Transition

Sections 4–9 of [ER60] are devoted to global properties of random graphs. The proofs follow the same pattern. First, the expectation of the quantity in question is asymptotically evaluated. Then, using Markov's and Chebyshev's inequality (the first and the second moment method, resp.) the asymptotics of the quantities themselves are derived. As a summary of these results we quote here how Erdős and Rényi characterize the process of the evolution of a random graph in the paper presented to the International Statistical Institute meeting in Tokyo in 1961 [ER61a]:

"*If n is fixed large positive integer and n is increasing from 1 to $\binom{n}{2}$, the evolution of $\Gamma_{n,N}$ passes through five clearly distinguishable phases. These phases correspond to ranges of growth of the number N of edges, these ranges being defined in terms of the number n of vertices.*

- **Phase 1** *corresponds to the range $N(n) = o(n)$. For this phase it is characteristic that $\Gamma_{n,N(n)}$ consists almost surely (i.e. with probability tending to 1 as $n \to +\infty$) exclusively of components which are trees. (...)*
- **Phase 2** *corresponds to the range $N(n) \sim cn$ with $0 < c < 1/2$. (...) In this range almost surely all components of $\Gamma_{n,N(n)}$ are either trees or components consisting of an equal number of edges and vertices, i.e. components containing exactly one cycle. (...) In this phase though not all, but still almost all (i.e. $n - o(n)$) vertices belong to components which*

are trees. The mean number of components is $n - N(n) + O(1)$*, i.e. in this range by adding a new edge the number of components decreases by* 1*, except for the finite number of steps.*

- **Phase 3** *corresponds to the range* $N(n) \sim cn$ *with* $c \geq 1/2$*. When* $N(n)$ *passes the threshold* $n/2$*, the structure of* $\Gamma_{n,N(n)}$ *changes abruptly. As a matter of fact this sudden change of the structure of* $\Gamma_{n,N(n)}$ *is the most surprising fact discovered by the investigation of the evolution of random graphs. While for* $N(n) \sim cn$ *with* $c < 1/2$ *the greatest component of* $\Gamma_{n,N(n)}$ *is a tree and has (with probability tending to* 1 *as* $n \to +\infty$*) approximately* $\frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right)$ *vertices, where* $\alpha = 2c - \log 2c$*, for* $N(n) \sim n/2$ *the greatest component has (with probability tending to* 1 *as* $n \to +\infty$*) approximately* $n^{2/3}$ *vertices and has rather complex structure. Moreover for* $N(n) \sim cn$ *with* $c > 1/2$ *the greatest component of* $\Gamma_{n,N(n)}$ *has (with probability tending to* 1 *as* $n \to +\infty$*) approximately* $G(c)n$ *vertices, where*

$$G(c) = 1 - \frac{1}{2c}\sum_{k=1}^{+\infty}\frac{k^{k-1}}{k!}\left(2ce^{-2c}\right)^k$$

(clearly $G(1/2) = 0$ *and* $\lim_{c\to+\infty} G(c) = 1$*).*

Except this "giant" component, the other components are all relatively small, most of them being trees, the total number of vertices belonging to components, which are trees being almost surely $n(1 - G(c)) + o(n)$ *for* $c \geq 1/2$*. (...)*

The evolution of $\Gamma_{n,N(n)}$ *in Phase 3. may be characterized by that the small components (most of which are trees) melt, each after another, into the giant component, the smaller components having the larger chance of "survival"; the survival time of a tree of order* k *which is present in* $\Gamma_{n,N(n)}$ *with* $N(n) \sim cn$*,* $c > 1/2$ *is approximately exponentially distributed with mean value* $n/2k$*.*

- **Phase 4** *corresponds to the range* $N(n) \sim cn\log n$ *with* $c \leq 1/2$*. In this phase the graph almost surely becomes connected. (...)*
- **Phase 5** *consists of range* $N(n) \sim (n\log n)\omega(n)$ *where* $\omega(n) \to +\infty$*. In this range the whole graph is not only almost surely connected, but the orders of all points are almost surely asymptotically equal. Thus the graph becomes in this phase "asymptotically regular"."*

Erdős and Rényi in their fundamental paper [ER60] gave a fairly complete "big picture" of the evolution of a random graphs. However many fascinating questions were left unanswered. For example, how did the giant component grow so rapidly, what is the nature of the "double jump" of its size: from $O(\log n)$ when $c < 1/2$ to $\Theta(n^{2/3})$ when $c = 1/2$ and finally being of the order of n when $c > 1/2$?

Often we say that a random graph goes through the phase transition at $c = 1/2$ due to an obvious resemblance of this period of its evolution to the physical phenomena of changing the state, for example, from liquid to solid. Here a random graph changes abruptly its state from a loose collection of small components being trees and unicyclic to solid single giant component dominating its structure.

The critical moment of the phase transition was unresolved until the milestone paper of Béla Bollobás [13] who revealed the mechanism of the formation of the giant component. He also focused the attention, for the first time, on the nature of the phase transition phenomena, investigating this critical moment of the evolution and looking at the beginning of so called *supercritical phase*. He asked what is the typical structure of a random graph $\Gamma_{n,N}$ when $N(n) = \frac{1}{2}n + s$, where $s = o(n)$. In particular he proved that the largest component is almost surely unique once $s \geq 2(\log n)^{1/2} n^{2/3}$ and its size $L_1(\Gamma_{n,N})$ is approximately $4s$ while the size of the second largest component $L_2(\Gamma_{n,N})$ is much smaller.

Bollobás gave a good lead to what we might consider as the proper magnification if we want to get undistorted picture of the phase transition while looking at the neighborhood of the "critical point" $n/2$. Due to later results of Łuczak [58], combined with those of Kolchin [51], we know that the correct parametrization is

$$N(n) = \frac{1}{2}n + \lambda n^{2/3}.$$

When $\lambda \to -\infty$ then $\Gamma_{n,N}$ consists of many components of the same size as the largest one, which is still very small and consists roughly of $\frac{n^2}{2s^2}\log(s^3/n^2)$ vertices, and the large components are unable to "swallow" each other and therefore are forced to hunt for smaller query. Hence large components grow absorbing only small ones and no clear favorite to win the race for the giant emerges. As the number of edges $N(n)$ increases, the number of contestants decreases. When $\lambda = \textit{constant} < 0$ the probability that two specified large components will form a new component is bounded away from zero, but still too small to ensure the creation of unique giant component. At the same time, a big gap between the orders of large and small components arises which prevents the creation of new large components from the small ones. Next, as soon as $\lambda \to \infty$, all large components almost "instantly" merge together and a unique large component emerges. This component is still not giant, it has barely over $n^{2/3}$ vertices, but it will continue to absorb other components, first the largest ones, rapidly becoming giant.

The next result of Łuczak [58] gives a clear picture of the sizes $L_i(\Gamma_{n,N})$ of the ith largest components during the phase transition of $\Gamma_{n,N}$. Here and throughout the paper the abbreviation a.s. stands for 'almost surely', a phrase whose precise meaning was explained in the description of Phase 1. above.

Theorem 4.1 ([**58**]). *Let k be natural number and $sn^{-2/3} \to \infty$ but $s = o(n)$.*

(i) If $N = n/2 - s$ then for every $i = 1, 2, \ldots, k$ and every real r

$$\lim_{n\to\infty} P\left(L_i(\Gamma_{n,N}) < \frac{n^2}{2s^2}\left(\log\frac{s^3}{n^2} - \frac{5}{2}\log\log\frac{s^3}{n^2} + r\right)\right) = \sum_{j=0}^{i-1}\frac{\lambda^j}{j!}e^{-\lambda},$$

where $\lambda = \lambda(r) = 2/\sqrt{\pi}e^{-r}$.

Moreover, a.s. the ith largest component of $\Gamma_{n,N}$ is a tree for $i = 1, 2, \ldots, k$ and $\Gamma_{n,N}$ contains no component with more edges than vertices.

(ii) Let $N = n/2 + s$ and let s' be the unique positive solution of the equation

$$\left(1 - \frac{2s'}{n}\right)e^{\frac{2s'}{n}} = \left(1 + \frac{2s'}{n}\right)e^{-\frac{2s'}{n}}.$$

Then a.s.

$$\left|L_1(\Gamma_{n,N}) - \frac{2(s+s')n}{n+2s}\right| < \omega(n)\frac{n}{\sqrt{s}}$$

and so

$$|L_1(\Gamma_{n,N}) - 4s| < \omega(n)\frac{n}{\sqrt{s}} + O\left(\frac{s^2}{n}\right).$$

Moreover, for every $i =, 2, \ldots, k$ and every real r

$$\lim_{n\to\infty} P\left(L_i(\Gamma_{n,N}) < \frac{n^2}{2s^2}\left(\log\frac{s^3}{n^2} - \frac{5}{2}\log\log\frac{s^3}{n^2} + r\right)\right) = \sum_{j=0}^{i-1}\frac{\lambda^j}{j!}e^{-\lambda},$$

where $\lambda = \lambda(r) = 2/\sqrt{\pi}e^{-r}$.

Furthermore a.s. the ith largest component of $\Gamma_{n,N}$, $i = 2, 3, \ldots, k$, is a tree and no component of $\Gamma_{n,N}$, except for the largest one, contains more edges than vertices.

To study the critical "interval" when the phase transition takes place, i.e., when $N(n) = \frac{1}{2}n + \lambda n^{2/3}$ and $\lambda \to \mp\infty$, requires very sophisticated and delicate tools. Janson, Knuth, Łuczak and Pittel in their extensive, almost 140 pages long, study [40] applied machinery of generating functions with great success. They were able to analyze the structure of evolving graphs (and multigraphs) when edges are added one at a time and at random, with great precision, mainly looking and so called excess and deficiency of a graph. To give the reader a taste of their results let us quote the following theorem.

Theorem 4.2 ([**40**]). *The probability that a random graph or multigraph with n vertices and $\frac{1}{2}n + O(n^{1/3})$ edges has exactly r bicyclic components (i.e., components with exactly two cycles), and no components of higher cyclic order, is*

$$\left(\frac{5}{18}\right)^r \sqrt{\frac{2}{3}} \frac{1}{(2r)!} + O(n^{-1/3}).$$

They also study the following fascinating problem: What is the probability that the component which during the evolution becomes the first "complex" component (i.e., the first component with more than one cycle) is the only complex component which emerges during the whole process? So they ask what is the probability that the first bicyclic component is the "seed" for the giant one. They prove that it happens quite often indeed.

Theorem 4.3 ([**40**]). *The probability that an evolving graph or multigraph on n vertices never has more than one complex component throughout its evolution approaches $\frac{5\pi}{18} \approx 0.8727$ as $n \to \infty$.*

5. Planarity and Chromatic Number

In a paper of such an enormous length one can likely find less rigorous claims. One of such things happened in the paper [ER60] in relation to the question when a random graph $\Gamma_{n,N}$ is planar.

Since trees and components with exactly one cycle are planar, Erdős and Rényi easily deduced from their findings about early stages of the evolution of a random graph, that when $c < 1/2$ then the probability that $\Gamma_{n,N}$ is planar tends to 1. Now, to support the claim that when c passes $1/2$ the graph becomes non-planar they used the argument that $\Gamma_{n,N}$ contains an induced cycle with d diagonals. Although their claim (Theorem 8a on page 51) regarding the distribution of the number of such cycles is incorrect, as it was pointed out later by Łuczak and Wierman [63], their intuition was perfect and the following result is indeed true.

Theorem 5.1 ([**63**]). *Let us suppose that $N \sim cn$. If $c < 1/2$ the probability that the graph $\Gamma_{n,N}$ is planar is tending to 1 while for $c > 1/2$ this probability tends to 0.*

Such a behavior of a random graph shows the fundamental difference in its typical structure before and after the phase transition. Now, thanks to the contribution of Łuczak, Pittel and Wierman [64], we have more detailed knowledge about planarity of a random graph, also during the phase transition.

Theorem 5.2 ([**64**]). *Let $\epsilon = \epsilon(n) \to 0$ as $n \to \infty$. Then $\Gamma_{n,p}$ is:*

(i) *a.s. planar, when* $p = (1-\epsilon)/n, \epsilon^3 n \to \infty$*;*
(ii) *Planar with probability tending to* $a(\lambda)$, $0 < a(\lambda) < 1$*, as* $n \to \infty$*, when* $p = (1+\epsilon)/n$*, where* $\epsilon^3 n \to \lambda$ *and* $-\infty < \lambda < \infty$ *is a constant;*
(iii) *a.s. non-planar, when* $p = (1+\epsilon)/n, \epsilon^3 n \to \infty$*.*

In the final section of the paper [ER60] Erdős and Rényi collected unsolved problems. One of them is closely related to planarity: *Another interesting question is: what is the threshold for the appearance of a "topological complete graph of order k", i.e., of k points such that any two of them can be connected by a path and these paths do not intersect. For* $k > 4$ *we do not know the solution.* The solution was found many years later by Ajtai, Kómlos, and Szemerédi [2].

Another problem mentioned there turned out to be one of the central and most challenging questions of the theory. Erdős and Rényi asked "*what will be the chromatic number of* $\Gamma_{n,N}$ *?*" What they knew then about this important graph invariant was limited to facts which can be deduced from general results regarding the evolutionary process. Here is what they were able to conclude : "*Clearly every tree can be colored by 2 colors, and thus by Theorem 4a almost surely* $Ch(\Gamma_{n,N}) = 2$ *if* $N(n) = o(n)$*. As however the chromatic number of a graph having an equal number of vertices and edges is equal to* 2 *or* 3 *according whether the only cycle contained in such graph is of even or odd order, it follows from Theorem 5e that almost surely* $Ch(\Gamma_{n,N}) \le 3$ *for* $N(n) \sim nc$ *with* $c < 1/2$*. For* $N(n) \sim n/2$ *we have almost surely* $Ch(\Gamma_{n,N}) \ge 3$*. As a matter of fact, in the same way, as we proved Theorem 5b, one can prove that* $\Gamma_{n,N}$ *contains for* $N(n) \sim n/2$ *almost surely a cycle of odd order. It is an open problem how large* $Ch(\Gamma_{n,N})$ *is for* $N(n) \sim n/2$ *with* $c > 1/2$*.*"

This question remained open for next 30 years, and was answered, for large c, by Łuczak in [57]. He proved that the chromatic number $\chi(\Gamma_{n,p})$ behaves as follows.

Theorem 5.3 ([**57**]). *Let* $np = c$ *and* $\epsilon > 0$ *be fixed. Suppose* $c_\epsilon \le c + o(n)$ *for sufficiently large constant* c_ϵ*. Then*

$$P\left(\frac{c}{2\log c} < \chi(\Gamma_{n,p}) < (1+\epsilon)\frac{c}{2\log c}\right) \to 1 \quad as \quad n \to \infty.$$

Although the original question was posed for sparse random graphs the ideas leading to the proof came from investigations of the chromatic number of dense random graphs. The first step toward the solution was made by Matula [66, 67] and Bollobás and Erdős [16] who discovered high concentration of the size of the largest independent set in $\Gamma_{n,p}$ around $2\log_b n$, where $b = 1/(1-p)$ and edge probability p is a constant. It suggested that the respective lower bound for $\chi(\Gamma_{n,p})$ should be $n/(2\log_b n)$. Only a few years later, Grimmett and McDiarmid published a paper [38] in which they showed that a greedy algorithm, which assigns colors to vertices of a random graph sequentially, in such a way that a vertex gets the first available color, needs, with high probability, approximately $n/\log_b n$ colors to produce a proper coloring of $\Gamma_{n,p}$. It established an upper bound for the chromatic number of dense random graph, twice as large as the lower bound. Grimmett and McDiarmid conjectured that the lower bound sets, in fact, the correct order of magnitude for $\chi(\Gamma_{n,p})$. The right tool to settle this conjecture was delivered by Shamir and Spencer [76]. They proved that the chromatic number of $\Gamma_{n,p}$ is sharply concentrated in an interval of length of order $n^{1/2}$ but, what perhaps was more important then their result itself, they introduced to the theory of random graphs a new powerful technique based on concentration measure of martingales, known in the probabilistic literature as Hoeffding-Azuma inequality. But it was Béla Bollobás who showed how the potential of martingale approach can be utilized to solve long standing conjecture. In his paper [15] he proved the following theorem.

Theorem 5.4 (**[15]**). *Let $0 < p < 1$ be fixed and $b = 1/(1-p)$. Then for every $\epsilon > 0$*

$$P\left(\frac{n}{2\log_b n} < \chi(\Gamma_{n,p}) < (1+\epsilon)\frac{n}{2\log_b n}\right) \to 1 \quad \textit{as} \quad n \to \infty.$$

Later on Matula and Kucera [68] gave an alternative proof of the above theorem, using the second moment and "expose and merge" algorithmic approach. Łuczak's proof of Theorem 5.3 is in fact an ingenious blend of the martingale and "expose and merge" techniques.

The chromatic number of a random graph is a random variable, the distribution of which should be highly concentrated. It is easy to notice (see above) that if $p = o(n^{-1})$ then $\chi(\Gamma_{n,p})$ is 2 (not counting the case when the edge probability is of the order smaller then n^{-2} and therefore, with high probability the graph is empty). One can also show that when $p \sim cn^{-1}$, $O < c < 1$ then $P(\chi(\Gamma_{n,p}) = 2) \to a$ and $P(\chi(\Gamma_{n,p}) = 3) \to 1 - a$, where $a = e^{c/2}((1-c)/(1+c))^{1/4}$. The last probabilities are simply the same as the probabilities that $\Gamma_{n,p}$ has or does not have an odd cycle. Such a behavior of a random variable χ has been confirmed, for small edge probabilities only, by Łuczak. He proved in [61] that if $p < n^{-5/6-\epsilon}$ then the chromatic number, as expected, takes on at most two values.

6. Asymmetric Graphs

Another interesting topic originated from a joint paper by Erdős and Rényi in the peak of their cooperation in early 1960s [ER63]. Here is how they describe their goals: "*We shall call (...) a graph* symmetric*, if there exists a non-identical permutation of its vertices, which leaves the graph invariant. By other words, a graph is called symmetric if the group of its automorphisms has degree greater than 1. A graph which is not symmetric will be called* *asymmetric. The degree of symmetry of a symmetric graph is evidently measured by the degree of its group of automorphisms. The question which led us to the results contained in the present paper is the following: how can we measure the degree of asymmetry of an asymmetric graph?*"

They answer the last question in what follows: "*Evidently any asymmetric graph can be made symmetric by deleting certain of its edges and by adding certain new edges connecting its vertices. We shall call such a transformation of the graph its symmetrization. For each symmetrization of the graph let us take the sum of the number of deleted edges – say r – and the number of new edges – say s –; it is reasonable to define the degree of asymmetry $A[G]$ of a graph G, as the minimum of $r+s$ where the minimum is taken over all possible symmetrizations of the graph G. $(\dots)$ The question arises: how large can be the degree of asymmetry of a graph of order n (i.e., a graph which has n vertices)? We shall denote by $A(n)$ the maximum of $A[G]$ for all graphs G of order $n(n = 2, 3, \dots)$.*"

They first notice that $A(2) = A(3) = A(4) = A(5) = 0$ while $A(6) = 1$. In general, a rather straightforward deterministic argument leads to the following result.

Theorem 6.1 ([**30**]).

$$A(n) \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

To find the lower bound for $A(n)$ Erdős and Rényi use a non-constructive argument, i.e., they show via the probabilistic method that there exists a certain graph on n vertices with the degree of asymmetry at least $n(1-\epsilon)/2$, $0 < \epsilon < 1$.

Theorem 6.2 ([**30**]). *Let us choose at random a graph Γ having n given vertices so that all possible $2^{\binom{n}{2}}$ graphs should have the same probability to be chosen. Let $\epsilon > 0$ be arbitrary. Let $P_n(\epsilon)$ denote the probability that by changing not more than $\frac{n(1-\epsilon)}{2}$ edges of Γ it can be transformed into a symmetric graph. Then we have*

$$\lim_{n\to\infty} P_n(\epsilon) = 0.$$

Corollary 6.1. *For any ϵ with $0 < \epsilon < 1$ there exists an integer $n_0(\epsilon)$ depending only on ϵ, such that for every $n > n_0(\epsilon)$ there exists a graph G of order n with $A[G] > n(1-\epsilon)/2$.*

Indeed, for large n, Theorem 6.2 shows that almost every graph is a counterexample to the hypothesis that its symmetrization is possible with less than $\frac{n}{2}(1 - o(1))$ edges.

Hence, if we combine Theorem 6.1 and Corollary 6.1 we see that

$$\lim_{n\to\infty} \frac{A(n)}{n} = \frac{1}{2}.$$

After showing that almost all labeled simple graphs are asymmetric, Erdős and Rényi turned their attention to graphs with a prescribed number of edges. First they noticed that since almost every tree has a cherry, i.e., a pair of pendant vertices adjacent to a common neighbor, therefore almost every tree on n vertices is symmetric. Furthermore they proved that any connected graph of order n having n edges is either symmetric or its asymmetry is one and gave the following bound.

Theorem 6.3 ([**30**]). *If a graph G of order n has $N = \lambda n$ edges $(0 < \lambda < (n-1)/2)$ then*

$$A[G] \le 4\lambda \left(1 - \frac{2\lambda}{n-1}\right).$$

Erdős and Rényi went further in their investigations. Let us quote a few more lines from their paper [ER63]. "*Another interesting question is to investigate the asymmetry or symmetry of a graph for which not only the number of vertices but also the number of edges N is fixed, and to ask that if we choose one of these graphs at random, what is the probability of its being asymmetric. We have solved this question too, and have shown that if $N = \frac{n}{2}(\log n + \omega(n))$, where $\omega(n)$ tends arbitrarily slowly to $+\infty$ for $n \to +\infty$, then the probability that a graph with n vertices and N edges chosen at random (so that any such graph has the same probability $\binom{\binom{n}{2}}{N}^{-1}$ to be chosen) should be asymmetric, tends to 1 for $n \to +\infty$. This and some further results will be published in another forthcoming paper.*"

Unfortunately the announced paper has never been published! Several years later this problem and the analogous one for unlabeled graphs was attacked again by Wright [79].

Consider graphs $\Gamma_{n,N}$ and $U_{n,N}$ picked at random from the families of all labeled and unlabeled graphs on n vertices and with $N = N(n)$ edges, respectively. Here is the result of Wright.

Theorem 6.4 ([**79**]). *If $\omega(n) = (2N(n)/n) - \log n \to \infty$ then $\Gamma_{n,N}$ and $U_{n,N}$ are almost surely asymmetric while when $\omega(n) \le 0$ then they are almost surely symmetric.*

Later Łuczak [56] gave precise results about the structure of the automorphism group $Aut(\Gamma_{n,N})$ of a random graph $\Gamma_{n,N}$. He studied the symmetry of the largest component $L_1(n, N)$ of this random graph. What he found was that when $N(n) = \frac{1}{2}n\alpha(n)$ then there exists a constant d such that for $\alpha(n) \geq d$ almost surely $Aut(L_1(n,N)$ is isomorphic to some product of symmetric groups. From this result he was able to deduce the following strengthening of the "labelled" part of Theorem 6.4.

Theorem 6.5 ([**56**]). *Let* $N = \frac{n}{2}(\log n + \omega(n))$.

(i) If $\omega(n) \to -\infty$ *then* $|Aut(\Gamma_{n,N})| \to \infty$ *a.s.*
(ii) If $\omega(n) \to c$ *then*

$$\lim_{n\to\infty} P(|Aut(\Gamma_{n,N})| = 1) = e^{\lambda}(1+\lambda)$$

$$\lim_{n\to\infty} P(|Aut(\Gamma_{n,N})| = k!) = \frac{\lambda^k}{k!}e^{-\lambda}$$

for $k = 2, 3, \ldots,$ *where* $\lambda = e^{-c}$ *and* c *is a constant.*
(iii) If $\omega(n) \to \infty$ *then* $|Aut(\Gamma_{n,N})| = 1$ *a.s.*

7. Perfect Matchings

The last three papers Erdős and Rényi wrote on the subject of random graphs were devoted to the existence of 1-factors. In [ER64] and [ER68] they coped with the relatively easier case of random bipartite graphs. In both papers they consequently emphasized the matrix terminology. "*In the present paper we deal with certain random* 0-1 *matrices. Let* $\mathcal{M}(n,N)$ *denote the set of all* n *by* n *square matrices among the elements of which there are exactly* N *elements* ($n \leq N \leq n^2$) *equal to* 1*, all the other elements are equal to* 0*. The set* $\mathcal{M}(n,N)$ *contains clearly* $\binom{n^2}{N}$ *such matrices; we consider a matrix* M *chosen at random from the set* $\mathcal{M}(n,N)$*, so that each element of* $\mathcal{M}(n,N)$ *has the same probability* $\binom{n^2}{N}^{-1}$ *to be chosen. We ask how large* N *has to be, for a given large value of* n*, in order that the permanent of the random matrix* M *should be different from zero with probability* $\geq \alpha$*, where* $0 < \alpha < 1$*. (...) A second way to formulate the problem is as follows: we shall say that two elements of a matrix are in independent position if they are not in the same row and not in the same column. Now our question is to determine the probability that the random matrix* M *should contain* n *elements which are all equal to 1 and pairwise in independent position.*"

The result they prove resembles that for the connectedness (compare Theorem 2.1).

Theorem 7.1 ([**31**]). *Let* $P(n,N)$ *denote the probability of the event that the permanent of the random matrix* M *is positive. Then if*

$$N(n) = n\log n + cn + o(n)$$

where c is any real constant, we have

$$\lim_{n\to\infty} P(n, N(n)) = e^{-2e^{-c}}.$$

Finally, they also mention graphs: "*This result can be interpreted also in the following way, in terms of graph theory. Let $\Gamma_{n,N}$ be a bichromatic random graph containing n red and n blue vertices, and N edges which are chosen at random among the n^2 possible edges connecting two vertices having different color (so that each of the $\binom{n^2}{N}$ possible choices has the same probability). Then $P(n, N)$ is equal to the probability that the random graph $\Gamma_{n,N}$ should contain a factor of degree 1, i.e., $\Gamma_{n,N}$ should have a subgraph which contains all vertices of $\Gamma_{n,N}$ and n disjoint edges, i.e., n edges which have no common endpoint.*" (They seem not to use the name 'perfect matching' at all.)

As far as the proof is concerned, "*Besides elementary combinatorial and probabilistic arguments similar to that used by us in our previous work on random graphs (...) our main tool in proving our results is the well-known theorem of D. König, which is nowadays well known in the theory of linear programming, according to which if M is an n by n matrix, every element of which is either 0 or 1, then the minimal number of lines (i.e., rows or columns) which contain all the 1-s, is equal to the maximal number of 1-s in independent position. As a matter of fact, for our purposes we need only the special case of this theorem, proved already by Frobenius (1917), concerning the case when the maximal number of ones in independent positions is equal to n (...). According to the theorem of Frobenius-König $1 - P(n, N)$ is equal to the probability that there exists a number k such that there can be found k rows and $n-k-1$ columns of M which contain all the ones $(0 \le k \le n-1)$.*" The rest of the proof is devoted to showing that this is very unlikely for $N(n)$ given. It is interesting to notice that Erdős and Rényi never mention Hall's theorem, which is equivalent to Frobenius but far more popular in combinatorics nowadays.

The 1968 paper is a straightforward extension of the 1964 result, where it is shown that setting

$$N(n) = n \log n + (r-1)n \log\log n + n\omega(n)$$

where $\omega(n)$ tends arbitrarily slowly to infinity then almost surely the bichromatic random graph contains r disjoint 1-factors. The only new element of the proof is the observation that if there are no r disjoint 1-factors then there is a way to delete some edges so that no vertex looses more than $r-1$ from its degree and the resulting subgraph contains no 1-factor at all. Then again the theorem of Frobenius is used.

The most involved of the three papers about 1-factors is that from 1966, where an ordinary (not bichromatic) random graph $\Gamma_{n,N}$ is considered. The reason is that the theorem of Tutte describing the structure of graphs which admit 1-factors is more complex than its counterpart in the bipartite case. "*It should be added that the problem investigated in the present paper is much more difficult than the corresponding problem for even graphs solved in [5]. Thus for instance in [5] we made use of the well known theorem of D. König; the corresponding tool in the present paper is the much deeper theorem of Tutte mentioned above.*" ([5] = [ER 64])

The result of that paper says that the threshold for containing a 1-factor coincides with that for disappearance of isolated vertices, and thus also with that for connectivity (see Theorem 2.1). The proof is long and tedious and involves a weaker version of Tutte's theorem ignoring the parity of components.

Erdős and Rényi make also the following claim: "*If $N = \frac{1}{2}n\log n + O(n)$, as mentioned above, with probability near to 1, $\Gamma_{n,N}$ consists of a connected component and a certain number of isolated points. With the same method (...) one can prove that if the connected component of $\Gamma_{n,N}$ consists of an even number of points, it has with probability near 1 a factor of degree one. As the proof of this result is almost the same (...) we do not go into the details.*"

The above mentioned result was proved (in a strengthened form) by Bollobás and Thomason [18]. In order to quote that result let us extend the notion of a perfect matching by saying that a graph satisfies property $\mathcal{PM}$ if there is a matching covering all but at most one of the nonisolated vertices. It is known that, switching to the binomial model, as soon as $2np - \log n - \log\log n \to \infty$, there are only isolated vertices outside the giant component. However, the main obstacle for the property $\mathcal{PM}$ is the presence of a pair (at least two such pairs when the number of nonisolates is odd) of vertices of degree 1 adjacent to the same vertex (called, as we already mentioned, 'a cherry'). The expected number of cherries is

$$3\binom{n}{3}p^2(1-p)^{2(n-3)} < n^3p^2e^{-2np+6p} = o(1)$$

if $2np - \log n - 2\log\log n \to \infty$. Again, a trivial necessary condition becomes almost surely sufficient.

Theorem 7.2 (**[18]**). *Let $y_n = 2np - \log n - 2\log\log n \to \infty$. Then*

$$P(\Gamma_{n,p} \in \mathcal{PM}) \to \begin{cases} 0 & \text{if } y_n \to -\infty \\ e^{-\frac{1}{8}e^{-c}} & \text{if } y_n \to c \\ 1 & \text{if } y_n \to \infty. \end{cases}$$

The proof, again, was based on Tutte's theorem. Years later Łuczak and Ruciński proposed an alternative approach, via Hall's Theorem, invented in [65] to attack a more general question. For a given graph G, a perfect G-matching of a graph is a spanning subgraph which is a disjoint union of copies of G. For $G = K_2$ this is the ordinary notion of a 1-factor.

In [65] it was shown that for every nontrivial tree T, the threshold is the same as that for disappearance of isolated vertices.

Theorem 7.3 ([**65**]). *For every tree T on t vertices and with at least one edge, assuming n is divisible by t,*

$$P(\Gamma_{n,p} \textit{ has a perfect } T\textit{-matching }) \to \begin{cases} 0 & \textit{if } np - \log n \to -\infty \\ e^{-e^{-c}} & \textit{if } np - \log n \to c \\ 1 & \textit{if } np - \log n \to \infty. \end{cases}$$

The threshold for arbitrary G is not known in general. Some partial results are contained in [4] and [72].

Coming back to the original papers of Erdős and Rényi, the last of them is concluded by the following problem: "*does a random graph $\Gamma_{n,N}$ where n is even and*

$$N = \frac{1}{2} n \log n + \frac{r-1}{2} n \log \log n + \omega(n) n$$

where $\omega(n) \to \infty$, contain at least r disjoint factors of degree one with probability tending to 1 *for $n \to \infty$?*"

Shamir and Upfal [77] answered this question in the positive. Given a map f of $V(G)$ into the set of non-negative integers, define an f-factor of G as a spanning subgraph of G in which the degree of vertex x is $f(x)$.

Theorem 7.4 ([**77**]). *If*

$$p = \frac{1}{n}(\log n + (r-1) \log \log n + \omega(n),$$

$r \geq 1$, $\lim_{n\to\infty} \omega(n) = \infty$ *and* $1 \leq f(x_i) \leq r$, $\sum_{i=1}^{n} f(x_i)$ *even, then* $\Gamma_{n,p}$ *has an f-factor, almost surely.*

Although f-factors are characterized by Tutte's theorem, Shamir and Upfal chose an alternative approach using an algorithmic technique (introduced to random graphs by Pósa) of augmentation of sub-factors by alternating paths. In fact, the answer to the last question of Erdős and Rényi does not follow directly from the above result (not every r-factor has a 1-factorization) but from the proof. In 1985 Bollobás and Frieze [17] strengthened this answer by proving that almost surely in the random graph process of adding edges one by one, as soon as the minimum degree becomes r, there are $\lfloor r/2 \rfloor$ disjoint hamiltonian cycles plus a disjoint perfect matching if r is odd.

The next problem we would like to mention cannot be directly attributed to Erdős and Rényi. Here is how Erdős describes their omission [3, Appendix B]. "*When Rényi and I developed our theory of random graphs, we thought of extending our study for hypergraphs. We mistakenly thought that* *all (or most) of the extensions would be routine and we completely overlooked the following beautiful question of Shamir. (...) Shamir asked how many triples must one choose on* $3n$ *elements so that with probability bounded away from zero one should get* n *vertex disjoint triples. Shamir proved that* $n^{3/2}$ *triples suffice, but the truth may very well be* $n^{1+\epsilon}$ *or even* $cn \log n$*. The reason for the difficulty is that Tutte's theorem seem to have no analogy for triple systems or more generally for hypergraphs.*" The result mentioned by Erdős belongs, in fact, to J. Schmidt-Pruzan and E. Shamir [75]. In 1995, Frieze and Janson in [35] pushed the bound down to $n^{4/3}$.

Fortunately, Erdős and Rényi did not overlook some other important problems which stimulated the research in the theory of random graphs over the years. One such problem was the threshold for existence of a Hamiltonian cycle in a random graph. They, in fact, asked only: *for what order of magnitude of* $N(n)$ *has* $\Gamma_{n,N(n)}$ *with probability tending to* 1 *a Hamilton-line (i.e., a path which passes through all vertices)*. This problem was first tried by Pósa [69] and Korshunov [55] and finally solved by Kómlos and Szemerédi [54] and, in a stronger form, by Bollobás [12]. They proved that the threshold for Hamiltonian cycle coincides with that of disappearance of all vertices of degree 0 and 1.

8. Update for the Second Edition

We wrote this paper back in 1995. In this second edition of the volume we decided to leave the original text intact except for a few obvious corrections and the proofs of Theorems 3.2 and 3.3 which have been deleted entirely. However, several new developments have occurred afterward. Here we would like to mention some of them along with a couple of earlier results omitted in the first edition. Needles to say, our choice is quite subjective. For more thorough treatment of random graphs we refer the reader to the monograph [43] published in 2000.

In relation to connectivity, one should note that an old result of Łuczak [60] states that the k-core of a random graph, for p large enough, is *a.s.* empty or k-connected. It implies that $\Gamma_{n,p}$ is a.s. $c(\Gamma_{n,p})$-connected for the ranges of N larger than those in Theorem 2.2.

In the domain of small subgraphs of random graphs there has been an intense study of the so called *upper tail* of the random variable X_G counting copies of a given graph G in $\Gamma_{n,p}$. As far as the *lower tail* is concerned, whose special case is the probability $P(X = 0)$ discussed briefly after Theorem 3.2, the asymptotic order of magnitude of the logarithm of $P(X \leq (1 - \epsilon)EX)$ has been determined in [39] to be $-\Phi_G$. The exponent in the upper tail, $P(X \geq (1 + \epsilon)EX)$, is of a smaller order of magnitude which is still to be determined. In [44] general lower and upper bounds were obtained which differ only by a logarithmic factor. Very recently DeMarco and Kahn [21] have found the right threshold for cliques and formulated the "right" conjecture for the general case.

In Sect. 5, the threshold for topological cliques found in [2] has been sharpened (see [62], a remark after Corollary 18). A significant result about the chromatic number of a random graph appeared in [1]. Achlioptas and Naor found therein an explicit two-point limiting distribution of $\chi(\Gamma_{n,p})$, where $p = d/n$, for every $d > 0$, strengthening a theorem from [61] mentioned at the end of Sect. 5.

The most acclaimed result in random graph theory which appeared after 1995 is, without doubt, a solution to the celebrated Shamir problem posed in Sect. 7. After some initial attempts (Krivelevich [52, 53] and Kim [50]), in 2008 Johansson, Kahn, and Vu [46] published a complete solution to both, the hypergraph Shamir problem and to its random graph counterpart (triangle-factors), receiving for their achievement the prestigious Fulkerson Prize. Quite recently in a series of papers, Dudek, Frieze, Loh, and Speiss [22–24, 34] obtained thresholds for the hamiltonicity of random uniform hypergraphs. In the hardest case of so called loose Hamilton cycles they incorporated in their proofs the result on perfect matchings from [46].

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