

# Refined Turán numbers and Ramsey numbers for the loose 3-uniform path of length three



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## ABSTRACT

Let  $P$  denote a 3-uniform hypergraph consisting of 7 vertices  $a, b, c, d, e, f, g$  and 3 edges  $\{a, b, c\}$ ,  $\{c, d, e\}$ , and  $\{e, f, g\}$ . It is known that the  $r$ -color Ramsey number for  $P$  is  $R(P; r) = r + 6$  for  $r \leq 7$ . The proof of this result relies on a careful analysis of the Turán numbers for  $P$ . In this paper, we refine this analysis further and compute, for all  $n$ , the third and fourth order Turán numbers for  $P$ . With the help of the former, we confirm the formula  $R(P; r) = r + 6$  for  $r \in \{8, 9\}$ .

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## 1. Introduction

For brevity, 3-uniform hypergraphs will be called here *3-graphs*. Given a family of 3-graphs  $\mathcal{F}$ , we say that a 3-graph  $H$  is  $\mathcal{F}$ -free if for all  $F \in \mathcal{F}$  we have  $H \not\supseteq F$ .

For a family of 3-graphs  $\mathcal{F}$  and an integer  $n \geq 1$ , the *Turán number of the 1st order*, that is, the ordinary Turán number, is defined as

$$\text{ex}^{(1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$

Every  $n$ -vertex  $\mathcal{F}$ -free 3-graph with  $\text{ex}^{(1)}(n; \mathcal{F})$  edges is called *1-extremal for  $\mathcal{F}$* . We denote by  $\text{Ex}^{(1)}(n; \mathcal{F})$  the family of all, pairwise non-isomorphic,  $n$ -vertex 3-graphs which are 1-extremal for  $\mathcal{F}$ . Further, for an integer  $s \geq 1$ , the *Turán number of the  $(s + 1)$ -st order* is defined as

$$\text{ex}^{(s+1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n, H \text{ is } \mathcal{F}\text{-free, and } \forall H' \in \text{Ex}^{(1)}(n; \mathcal{F}) \cup \dots \cup \text{Ex}^{(s)}(n; \mathcal{F}), H \not\subseteq H'\},$$

if such a 3-graph  $H$  exists. Note that if  $\text{ex}^{(s+1)}(n; \mathcal{F})$  exists then, by definition,

$$\text{ex}^{(s+1)}(n; \mathcal{F}) < \text{ex}^{(s)}(n; \mathcal{F}). \quad (1)$$

An  $n$ -vertex  $\mathcal{F}$ -free 3-graph  $H$  is called  *$(s + 1)$ -extremal for  $\mathcal{F}$*  if  $|E(H)| = \text{ex}^{(s+1)}(n; \mathcal{F})$  and  $\forall H' \in \text{Ex}^{(1)}(n; \mathcal{F}) \cup \dots \cup \text{Ex}^{(s)}(n; \mathcal{F}), H \not\subseteq H'$ ; we denote by  $\text{Ex}^{(s+1)}(n; \mathcal{F})$  the family of  $n$ -vertex 3-graphs which are  $(s + 1)$ -extremal for  $\mathcal{F}$ . In the case when  $\mathcal{F} = \{F\}$ , we will write  $F$  instead of  $\{F\}$ .

A *loose 3-uniform path of length 3* is a 3-graph  $P$  consisting of 7 vertices, say,  $a, b, c, d, e, f, g$ , and 3 edges  $\{a, b, c\}$ ,  $\{c, d, e\}$ , and  $\{e, f, g\}$ . The *Ramsey number*  $R(P; r)$  is the least integer  $n$  such that every  $r$ -coloring of the edges of the complete 3-graph  $K_n$  results in a monochromatic copy of  $P$ . Gyárfás and Raesi [6] proved, among many other results, that  $R(P; 2) = 8$ . (This result was later extended to loose paths of arbitrary lengths, but still  $r = 2$ , in [13].) Then Jackowska [9] showed that  $R(P; 3) = 9$  and  $r + 6 \leq R(P; r)$  for all  $r \geq 3$ . In turn, in [10] and [11], Turán numbers of the first and second order,  $\text{ex}^{(1)}(n; P)$

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and  $\text{ex}^{(2)}(n; P)$ , have been determined for all feasible values of  $n$ , as well as the single third order Turán number  $\text{ex}^{(3)}(12; P)$ . Using these numbers, in [11], we were able to compute the Ramsey numbers  $R(P; r)$  for  $r = 4, 5, 6, 7$ .

**Theorem 1** ([6,9,11]). For all  $r \leq 7$ ,  $R(P; r) = r + 6$ .

In this paper we determine, for all  $n \geq 7$ , the Turán numbers for  $P$  of the third and the fourth order,  $\text{ex}^{(3)}(n; P)$  and  $\text{ex}^{(4)}(n; P)$ . The former allows us to compute two more Ramsey numbers.

**Theorem 2.** For all  $r \leq 9$ ,  $R(P; r) = r + 6$ .

It seems that in order to make a further progress in computing the Ramsey numbers  $R(P; r)$ ,  $r \geq 10$ , one would need to determine higher order Turán numbers  $\text{ex}^{(s)}(n; P)$ , at least for some small values of  $n$ . Unfortunately, the fourth order numbers are not good enough.

Throughout, we denote by  $S_n$  the 3-graph on  $n$  vertices and with  $\binom{n-1}{2}$  edges, in which one vertex, referred to as *the center*, forms edges with all pairs of the remaining vertices. Every sub-3-graph of  $S_n$  without isolated vertices is called *a star*, while  $S_n$  itself is called *the full star*. We denote by  $C$  *the triangle*, that is, a 3-graph with six vertices  $a, b, c, d, e, f$  and three edges  $\{a, b, c\}$ ,  $\{c, d, e\}$ , and  $\{e, f, a\}$ . Finally,  $M$  stands for a pair of disjoint edges.

In the next section we state all, known and new, results on ordinary and higher order Turán numbers for  $P$ , including Theorem 9 which provides a complete formula for  $\text{ex}^{(3)}(n; P)$ . We also define conditional Turán numbers and quote from [11] three useful lemmas about the conditional Turán numbers with respect to  $P, C, M$ . Then, in Section 3, we prove Theorem 2, while the remaining sections are devoted to proving Theorem 9.

**2. Turán numbers**

A celebrated result of Erdős, Ko, and Rado [2] asserts that for  $n \geq 6$ ,  $\text{ex}^{(1)}(n; M) = \binom{n-1}{2}$ . Moreover, for  $n \geq 7$ ,  $\text{Ex}^{(1)}(n; M) = \{S_n\}$ . We will need the second order version of this Turán number, together with the 2-extremal family. Such a result has been proved already by Hilton and Milner [8, Theorem 3,  $s = 1$ ] (see [4] for a simple proof). For a given set of vertices  $V$ , with  $|V| = n \geq 7$ , let us define two special 3-graphs. Let  $x, y, z, v \in V$  be four different vertices of  $V$ . We set

$$G_1(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : v \in h, h \cap \{x, y, z\} \neq \emptyset \right\},$$

$$G_2(n) = \{\{x, y, z\}\} \cup \left\{ h \in \binom{V}{3} : |h \cap \{x, y, z\}| = 2 \right\}.$$

Note that for  $i \in \{1, 2\}$ ,  $G_i(n) \not\supseteq M$  and  $|G_i(n)| = 3n - 8$ .

**Theorem 3** ([8]). For  $n \geq 7$ ,  $\text{ex}^{(2)}(n; M) = 3n - 8$  and  $\text{Ex}^{(2)}(n; M) = \{G_1(n), G_2(n)\}$ .

Later, we will also use the fact that  $C \subset G_i(n) \not\supseteq P$ ,  $i = 1, 2$ .

Recently, the third order Turán number for  $M$  has been established by Han and Kohayakawa. Let  $G_3(n)$  be the 3-graph on  $n$  vertices, with distinguished vertices  $x, y_1, y_2, z_1, z_2$  whose edge set consists of all edges spanned by  $x, y_1, y_2, z_1, z_2$  except for  $\{y_1, y_2, z_i\}$ ,  $i = 1, 2$ , and all edges of the form  $\{x, z_i, v\}$ ,  $i = 1, 2$ , where  $v \notin \{x, y_1, y_2, z_1, z_2\}$ . Note that  $|G_3(n)| = 8 + 2(n - 5) = 2n - 2$ .

**Theorem 4** ([7, Theorem 1.6]). For  $n \geq 7$ ,  $\text{ex}^{(3)}(n; M) = 2n - 2$  and  $\text{Ex}^{(3)}(n; M) = \{G_3(n)\}$ .

Interestingly, the number  $\binom{n-1}{2}$  serves as the Turán number for two other 3-graphs,  $C$  and  $P$ . The Turán number  $\text{ex}^{(1)}(n; C)$  has been determined in [3] for  $n \geq 75$  and later for all  $n$  in [1].

**Theorem 5** ([1]). For  $n \geq 6$ ,  $\text{ex}^{(1)}(n; C) = \binom{n-1}{2}$ . Moreover, for  $n \geq 8$ ,  $\text{Ex}^{(1)}(n; C) = \{S_n\}$ .

For large  $n$ , the Turán numbers for longer (than three) loose 3-uniform paths were found in [12]. The case of length three has been omitted in [12], probably because the authors thought it had been taken care of in [5], where  $k$ -uniform loose paths were considered,  $k \geq 4$ . However, the method used in [5] did not quite work for 3-graphs. In [10] we fixed this omission. Given two 3-graphs  $F_1$  and  $F_2$ , by  $F_1 \cup F_2$  denote a vertex-disjoint union of  $F_1$  and  $F_2$ . If  $F_1 = F_2 = F$  we will sometimes write  $2F$  instead of  $F \cup F$ .

**Theorem 6** ([10]).

$$\text{ex}^{(1)}(n; P) = \begin{cases} \binom{n}{3} & \text{and } \text{Ex}^{(1)}(n; P) = \{K_n\} & \text{for } n \leq 6, \\ 20 & \text{and } \text{Ex}^{(1)}(n; P) = \{K_6 \cup K_1\} & \text{for } n = 7, \\ \binom{n-1}{2} & \text{and } \text{Ex}^{(1)}(n; P) = \{S_n\} & \text{for } n \geq 8. \end{cases}$$

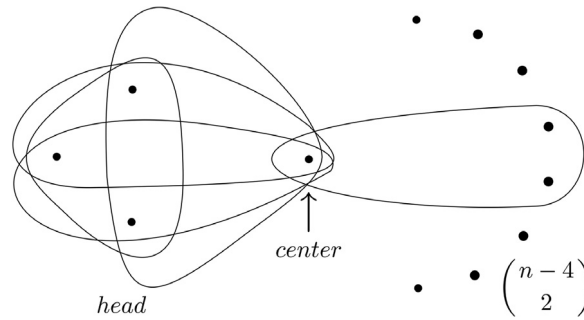


Fig. 1. The comet  $\text{Co}(n)$ .

Interestingly, although the ordinary Turán numbers for the 2-matching  $M$  and the 3-path  $P$  are equal for  $n \geq 8$ , their higher order counterparts differ significantly, being, respectively, of linear and quadratic order in  $n$ . In [11] we have completely determined the second order Turán number  $\text{ex}^{(2)}(n; P)$ , together with the corresponding 2-extremal 3-graphs. A comet  $\text{Co}(n)$  is an  $n$ -vertex 3-graph consisting of the complete 3-graph  $K_4$  and the full star  $S_{n-3}$ , sharing exactly one vertex which is the center of the star (see Fig. 1). This vertex is called *the center* of the comet, while the set of the remaining three vertices of the  $K_4$  is called its *head*.

**Theorem 7 ([11]).**

$$\text{ex}^{(2)}(n; P) = \begin{cases} 15 & \text{and } \text{Ex}^{(2)}(n; P) = \{S_7\} & \text{for } n = 7, \\ 20 + \binom{n-6}{3} & \text{and } \text{Ex}^{(2)}(n; P) = \{K_6 \cup K_{n-6}\} & \text{for } 8 \leq n \leq 12, \\ 40 & \text{and } \text{Ex}^{(2)}(n; P) = \{2K_6 \cup K_1, \text{Co}(13)\} & \text{for } n = 13, \\ 4 + \binom{n-4}{2} & \text{and } \text{Ex}^{(2)}(n; P) = \{\text{Co}(n)\} & \text{for } n \geq 14. \end{cases}$$

Note that for  $n \leq 6$  the second order number is not defined, since each 3-graph is a sub-3-graph of  $K_n$ . The main message behind the above result is that for  $n \geq 8$  it provides an upper bound on the number of edges in an  $n$ -vertex  $P$ -free 3-graph which is not a star.

Also in [11], we managed to calculate the third order Turán number for  $P$  and  $n = 12$ .

**Theorem 8 ([11]).**

$$\text{ex}^{(3)}(12; P) = 32 \quad \text{and} \quad \text{Ex}^{(3)}(12; P) = \{\text{Co}(12)\}.$$

The main Turán-type result of this paper provides a complete formula for the third order Turán number for  $P$ .

**Theorem 9.**

$$\text{ex}^{(3)}(n; P) = \begin{cases} 3n - 8 & \text{and } \text{Ex}^{(3)}(n; P) = \{G_1(n), G_2(n)\} & \text{for } 7 \leq n \leq 10, \\ 25 & \text{and } \text{Ex}^{(3)}(n; P) = \{G_1(n), G_2(n), \text{Co}(n)\} & \text{for } n = 11, \\ 32 & \text{and } \text{Ex}^{(3)}(n; P) = \{\text{Co}(n)\} & \text{for } n = 12, \\ 20 + \binom{n-7}{2} & \text{and } \text{Ex}^{(3)}(n; P) = \{K_6 \cup S_{n-6}\} & \text{for } 13 \leq n \leq 14, \\ 4 + \binom{n-5}{2} & \text{and } \text{Ex}^{(3)}(n; P) = \{K_4 \cup S_{n-4}\} & \text{for } n \geq 15. \end{cases}$$

In particular, for  $n \geq 14$ , Theorem 9 gives an upper bound on the number of edges in an  $n$ -vertex  $P$ -free 3-graph which is neither a star nor is contained in the comet  $\text{Co}(n)$ .

Surprisingly, as an immediate consequence we obtain an exact formula for the 4th Turán number for  $P$ , at least for  $n \geq 15$ . Indeed, consider the 3-graph  $\text{Ro}(n)$ , called *rocket*, obtained from the star  $S_{n-4}$  with center  $x$  by adding to it 4 more vertices, say,  $a, b, c, d$ , and three edges:  $\{x, a, b\}, \{a, b, c\}, \{a, b, d\}$ . Clearly,  $|\text{Ro}(n)| = |K_4 \cup S_{n-4}| - 1$ ,  $\text{Ro}(n) \not\subset S_n$ ,  $\text{Ro}(n) \not\subset \text{Co}(n)$ , and  $\text{Ro}(n) \not\subset K_4 \cup S_{n-4}$ . Hence,  $\text{ex}^{(4)}(n; P) \geq \text{ex}^{(3)}(n; P) - 1$ , but, in view of inequality (1), the two numbers cannot be equal.

In a similar fashion, by choosing an appropriate 4-extremal 3-graph, one can show that  $\text{ex}^{(4)}(7; P) = \text{ex}^{(3)}(7; P) - 1$  and  $\text{ex}^{(4)}(14; P) = \text{ex}^{(3)}(14; P) - 1$ . With some additional effort we were also able to calculate the remaining values of  $\text{ex}^{(4)}(n; P)$  and determine the families  $\text{Ex}^{(4)}(n; P)$  of 4-extremal 3-graphs. As these numbers are, however, of no use for calculating the respective Ramsey numbers, we state Theorem 10 without proof. Let  $K_5^{+2}$  be the 3-graph obtained from  $K_5$  by fixing two of its vertices, say  $a, b$ , and adding two more vertices  $c, d$  and two edges  $\{a, b, c\}$  and  $\{a, b, d\}$ .

**Theorem 10.**

$$\text{ex}^{(4)}(n; P) = \begin{cases} 12 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_3(n), K_5^{+2}\} & \text{for } n = 7, \\ 2n - 2 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_3(n)\} & \text{for } 8 \leq n \leq 9, \\ 20 & \text{and } \text{Ex}^{(4)}(n; P) = \{K_5 \cup K_5\} & \text{for } n = 10, \\ 20 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_3(n)\} & \text{for } n = 11, \\ 28 & \text{and } \text{Ex}^{(4)}(n; P) = \{G_1(n), G_2(n)\} & \text{for } n = 12, \\ 33 & \text{and } \text{Ex}^{(4)}(n; P) = \{K_6 \cup G_1(n), K_6 \cup G_2(n)\} & \text{for } n = 13, \\ 40 & \text{and } \text{Ex}^{(4)}(n; P) = \{2K_6 \cup 2K_1, K_4 \cup S_{10}\} & \text{for } n = 14, \\ 48 & \text{and } \text{Ex}^{(4)}(n; P) = \{\text{Ro}(n), K_6 \cup S_9\} & \text{for } n = 15, \\ 3 + \binom{n-5}{2} & \text{and } \text{Ex}^{(4)}(n; P) = \{\text{Ro}(n)\} & \text{for } n \geq 16. \end{cases}$$

To determine Turán numbers, it is sometimes useful to rely on [Theorem 3](#) and divide all 3-graphs into those which contain  $M$  and those which do not. To this end, it is convenient to define conditional Turán numbers (see [[10,11](#)]). For a family of 3-graphs  $\mathcal{F}$ , an  $\mathcal{F}$ -free 3-graph  $G$ , and an integer  $n \geq |V(G)|$ , the *conditional Turán number* is defined as

$$\text{ex}(n; \mathcal{F}|G) = \max\{|E(H)| : |V(H)| = n, H \text{ is } \mathcal{F}\text{-free, and } H \supseteq G\}.$$

Every  $n$ -vertex  $\mathcal{F}$ -free 3-graph  $H$  with  $\text{ex}(n; \mathcal{F}|G)$  edges and such that  $H \supseteq G$  is called  $G$ -*extremal* for  $\mathcal{F}$ . We denote by  $\text{Ex}(n; \mathcal{F}|G)$  the family of all  $n$ -vertex 3-graphs which are  $G$ -extremal for  $\mathcal{F}$ . (If  $\mathcal{F} = \{F\}$ , we simply write  $F$  instead of  $\{F\}$ .)

To illustrate the above mentioned technique, observe that for  $n \geq 7$

$$\text{ex}^{(2)}(n; P) = \max\{\text{ex}(n; P|M), \text{ex}^{(2)}(n; M)\} \stackrel{\text{Thm 3}}{=} \max\{\text{ex}(n; P|M), 3n - 8\} = \text{ex}(n; P|M),$$

the last equality holding for sufficiently large  $n$  (see [[11](#), Proof of Theorem 6] for details).

In the proof of [Theorem 9](#) we will use the following three lemmas, all proved in [[11](#)]. For the first one we need one more piece of notation. If, in the above definition, we restrict ourselves to connected 3-graphs only (connected in the weakest, obvious sense) then the corresponding conditional Turán number and the extremal family are denoted by  $\text{ex}_{\text{conn}}(n; \mathcal{F}|G)$  and  $\text{Ex}_{\text{conn}}(n; \mathcal{F}|G)$ , respectively.

**Lemma 1** ([[11](#), Lemma 1]). For  $n \geq 7$ ,

$$\text{ex}_{\text{conn}}(n; P|C) = 3n - 8 \quad \text{and} \quad \text{Ex}_{\text{conn}}(n; P|C) = \{G_1(n), G_2(n)\}.$$

[Lemma 1](#) as stated in [[11](#)] does not provide family  $\text{Ex}_{\text{conn}}(n; P|C)$ . However, it is clear from its proof that the  $C$ -extremal 3-graphs are the same as in [Theorem 3](#).

**Lemma 2** ([[11](#), Theorem 10]).

$$\text{ex}(n; \{P, C\}|M) = \begin{cases} 2n - 4 & \text{for } 6 \leq n \leq 9, \\ 20 & \text{for } n = 10, \\ 4 + \binom{n-4}{2} & \text{and } \text{Ex}(n; \{P, C\}|M) = \{\text{Co}(n)\} \quad \text{for } n \geq 11. \end{cases}$$

**Lemma 3** ([[11](#), Lemma 2]). For  $n \geq 6$

$$\text{ex}(n; \{P, C, P_2 \cup K_3\}|M) = 2n - 4,$$

where  $P_2$  is a pair of edges sharing one vertex.

**3. Proof of Theorem 2**

As mentioned in the Introduction, the inequality  $R(P; r) \geq r + 6, r \geq 1$ , has been already observed in [[9](#)]. We are going to show that  $R(P; r) \leq r + 6$  for  $8 \leq r \leq 9$ . Along the way, we need to strengthen the results for  $3 \leq r \leq 7$  as follows. Let  $K_n - e$  denote the 3-graph with  $n$  vertices and  $\binom{n}{3} - 1$  edges, while  $K_n - 2e$  denote each of the three possible (up to isomorphism) 3-graphs with  $n$  vertices and  $\binom{n}{3} - 2$  edges. Write  $H \rightarrow (P; r)$  if every  $r$ -coloring of the edges of  $H$  yields a monochromatic copy of  $P$ .

**Lemma 4.** For every  $n = 9, \dots, 13, K_n - 2e \rightarrow (P; n - 6)$ .

**Proof.** A coloring which does not yield a monochromatic copy of  $P$  is referred to as *proper*. Below, for each  $n$  we assume that there is a proper coloring of  $K_n - 2e$  and arrive at a contradiction.

**n = 9:** For every 3-coloring of  $K_9 - 2e$  there is a color with at least 28 edges and thus, if it is proper, then, by [Theorem 6](#), that color must form a full star. After removing the center of that star, we obtain a proper 2-coloring of  $K_8 - 2e$ , which, again by [Theorem 6](#), contains a monochromatic copy of  $P$ , a contradiction.

**n = 10:** For every 4-coloring of  $K_{10} - 2e$  there is a color with at least 30 edges and thus, if it is proper, then, by [Theorems 6](#) and [7](#), that color must form a star. After removing the center of that star we are back to the  $n = 9$  case.

**n = 11:** For every 5-coloring of  $K_{11} - 2e$  there is a color with at least 33 edges and thus, if it is proper, then, again by [Theorems 6](#) and [7](#), that color must form a star. After removing the center of that star we are back to the  $n = 10$  case.

We leave the most difficult case of  $n = 12$  to the end of the proof.

**n = 13:** For every 7-coloring of  $K_{13} - 2e$  there is a color with at least 41 edges and thus, if it is proper, then, one more time by [Theorems 6](#) and [7](#), that color must form a star. After removing the center of that star we jump to the  $n = 12$  case.

**n = 12:** Consider a 6-coloring of  $K_{12} - 2e$ . If a color forms a star, then, after removing its center, we obtain a 5-coloring of  $K_{11} - 2e$  (or  $K_{11} - e$  or  $K_{11}$ ) which, as we have already proved, contains a monochromatic copy of  $P$ . Thus, from now on we assume no color class forms a star. Consequently, by [Theorems 6–8](#), every color class with more than 32 edges is a sub-3-graph of  $K_6 \cup K_6$ . Since there is a color with at least 37 edges, it must be, as explained above, contained in a copy of  $K_6 \cup K_6$ . After removing that copy of  $K_6 \cup K_6$ , we are looking at a proper 5-coloring of a complete, 6 by 6 bipartite 3-graph  $H$ , with possibly up to two edges missing. As  $|H| \geq 220 - 40 - 2 = 178$ , the average number of edges per color is at least 35.6. On the other hand no color may have been applied to more than 36 edges. The reason is that, as explained above, such a color class would need to be a sub-3-graph of a copy of  $K_6 \cup K_6$ , but it is easy to check that every copy of  $K_6 \cup K_6$  shares at most 36 edges with  $H$ . This implies that among the five color classes at least three have size 36. But it was shown already in [[11](#)] (proof of [Theorem 1](#), case  $r = 6$ ) that the coexistence of three disjoint sub-3-graphs in  $H$ , each having 36 edges and contained in a copy of  $K_6 \cup K_6$ , is impossible.  $\square$

**Proof of [Theorem 2](#).** In the case  $r = 8$  we are going to prove a little stronger result to be used in the case  $r = 9$ .

**r = 8:** For every 8-coloring of  $K_{14} - e$  there is a color with at least 46 edges, and thus, if it is proper, then, by [Theorems 6, 7](#), and [9](#), that color must either form a star or be a sub-3-graph of the comet  $\text{Co}(14)$ . In either case, we remove the center of the structure in question, a star or a comet, and in addition, if it is the comet, the edge spanned by its head. We get a 7-coloring of  $K_{13} - e$  or  $K_{13} - 2e$  which, by [Lemma 4](#), yields a monochromatic copy of  $P$ , a contradiction. It follows that  $K_{14} - e \rightarrow (P; 8)$ , and so,  $K_{14} \rightarrow (P; 8)$  too. Hence, we have proved [Theorem 2](#) for  $r = 8$ .

**r = 9:** For every 9-coloring of  $K_{15}$  there is a color with at least 51 edges, and thus, if it is proper, then, again by [Theorems 6, 7](#), and [9](#), that color must form a sub-3-graph of  $S_{15}$  or  $\text{Co}(15)$ . Similarly to the case  $r = 8$ , we remove a vertex and possibly an edge, to obtain an 8-coloring of  $K_{14}$  or  $K_{14} - e$ , which, by the case  $r = 8$  yields a monochromatic copy of  $P$ , a contradiction.  $\square$

#### 4. Proof of [Theorem 9](#)

For  $n = 12$ , [Theorem 9](#) has been already proved in [[11](#)] (cf. [Theorem 8](#) in Introduction). Therefore, it seems natural to divide the proof into two ranges of  $n$ : smaller than 12 and larger than 12. The general set-up is, however, the same for both. We first check that all candidates for being 3-extremal 3-graphs do qualify, that is, are  $P$ -free, are not contained in any of the 1-extremal or 2-extremal 3-graphs with the same number of vertices, and have the number of edges given by the formula to be proved. Then, we consider an arbitrary  $n$ -vertex, qualifying 3-graph  $H$  and show that unless it is one of the candidate 3-extremal 3-graphs itself, its number of edges is strictly smaller than theirs.

For the latter task, we distinguish two cases: when  $H$  is connected and disconnected. The entire proof is inductive, in the sense that here and there we apply the very [Theorem 9](#) for smaller instances of  $n$ , once they have been confirmed.

##### 4.1. $7 \leq n \leq 11$

First note that by [Theorems 6](#) and [7](#), for  $7 \leq n \leq 11$

$$\text{Ex}^{(1)}(n; P) \cup \text{Ex}^{(2)}(n; P) = \{S_n, K_6 \cup K_{n-6}\}.$$

Moreover, for  $i \in \{1, 2\}$ ,  $G_i(n) \not\subseteq S_n$  and  $G_i(n) \not\subseteq K_6 \cup K_{n-6}$ . Consequently, since  $G_i(n)$  is  $P$ -free,

$$\text{ex}^{(3)}(n; P) \geq |G_i(n)| = 3n - 8.$$

We are going to show that, in fact, the 3-graphs  $G_i(n)$ ,  $i \in \{1, 2\}$  are the only 3-extremal 3-graphs for  $n \leq 10$ , whereas, for  $n = 11$ , in addition, the comet  $\text{Co}(11) \in \text{Ex}^{(3)}(11; P)$ .

For  $7 \leq n \leq 11$ , let  $H$  be an  $n$ -vertex  $P$ -free 3-graph, satisfying

$$H \not\subseteq S_n \quad \text{and} \quad H \not\subseteq K_6 \cup K_{n-6}. \tag{2}$$

We first assume that  $H$  is connected. If  $H \supset C$  or  $H \not\supset M$  then, by, respectively, [Lemma 1](#) or [Theorem 3](#),

$$|H| \leq 3n - 8,$$

with the equality for  $H = G_i(n)$ ,  $i \in \{1, 2\}$  only. Otherwise,

$$|H| \leq \text{ex}(n; \{P, C\}|M) \leq 3n - 8,$$

where the second inequality holds by Lemma 2, and it becomes equality only if  $n = 11$  and  $H = \text{Co}(11)$ .

Next, we show that if a  $P$ -free 3-graph  $H$  satisfying (2) is *disconnected*, then  $|H| < 3n - 8$ . Let  $m = m(H)$  be the number of vertices in the smallest component of  $H$ . Since  $n \leq 11$ , we have  $m \leq 5$ . Moreover,  $m \neq 2$ , since no component of a 3-graph may have two vertices. Thus,  $m \in \{1, 3, 4, 5\}$ . Note also that, as a consequence of the second part of (2), no union of components of  $H$  may have 6 vertices together. Consequently,  $m \neq n - 6$ . We now break the proof into several cases.

If  $n = 7$ , we must have  $m = 3$  and thus,

$$|H| \leq 1 + 4 < 3n - 8 = 13.$$

For  $n \geq 8$ , if  $m = 1$ , that is, if there is an isolated vertex  $v$  in  $H$ , then  $H - v$  still satisfies (2) with  $n - 1$  instead of  $n$ , and, by induction,

$$|H| \leq \text{ex}^{(3)}(n - 1; P) \leq 3(n - 1) - 8 < 3n - 8.$$

If  $m = 3$ , then, for  $n = 8, 10, 11$ , we apply the bound

$$|H| \leq 1 + \text{ex}^{(1)}(n - 3; P) < 3n - 8,$$

where the last inequality follows by Theorem 6. If  $m = 4$ , we have, similarly,

$$|H| \leq 4 + \text{ex}^{(1)}(n - 4; P) < 3n - 8,$$

for  $n = 8, 9, 11$ . Finally, if  $m = 5$  (and  $n = 10$ ),  $|H| \leq 2 \times \binom{5}{3} < 22$ .

#### 4.2. $n \geq 13$

By Theorems 6 and 7,

$$\text{Ex}^{(1)}(13; P) \cup \text{Ex}^{(2)}(13; P) = \{S_{13}, \text{Co}(13), K_6 \cup K_6 \cup K_1\},$$

while for  $n \geq 14$ ,

$$\text{Ex}^{(1)}(n; P) \cup \text{Ex}^{(2)}(n; P) = \{S_n, \text{Co}(n)\}.$$

Therefore, to determine  $\text{ex}^{(3)}(n; P)$  for  $n \geq 13$  we have to find the largest number of edges in an  $n$ -vertex  $P$ -free 3-graph  $H$  such that  $H \not\subseteq S_n, H \not\subseteq \text{Co}(n)$  and for  $n = 13$ , in addition,  $H \not\subseteq K_6 \cup K_6 \cup K_1$ . The 3-graphs

$$H_n := K_6 \cup S_{n-6} \quad \text{for } n \in \{13, 14\} \quad \text{and} \quad H_n := K_4 \cup S_{n-4} \quad \text{for } n \geq 15$$

satisfy all the above conditions. Hence, for  $n \geq 13$ ,

$$\text{ex}^{(3)}(n; P) \geq |H_n|.$$

We are going to show that also the opposite inequality holds, as well as, that the equality holds for  $H_n$  only.

To this end, let  $H \neq H_n$  be an  $n$ -vertex  $P$ -free 3-graph such that

$$H \not\subseteq S_n, \quad \text{and} \quad H \not\subseteq \text{Co}(n), \tag{3}$$

and for  $n = 13$ , in addition,  $H \not\subseteq K_6 \cup K_6 \cup K_1$ . We will show that  $|H| < |H_n|$ .

Assume first that  $H$  is *connected*. If, in addition,  $H \supset C$  or  $H \not\supset M$ , then, by, respectively, Lemma 1 or Theorem 3,

$$|H| \leq 3n - 8 < |H_n|.$$

Otherwise,  $H$  is a  $\{P, C\}$ -free, connected 3-graph containing  $M$ . Since, by Lemma 3,

$$\text{ex}(n; \{P, C, P_2 \cup K_3\}|M) = 2n - 4 < |H_n|,$$

we may assume that  $P_2 \cup K_3 \subset H$ . Thus, the connected case will be completed once we have proved the following lemma.

**Lemma 5.** For  $n \geq 13$ , if  $H$  is an  $n$ -vertex, connected,  $\{P, C\}$ -free 3-graph such that  $H \supset P_2 \cup K_3$  and  $H \not\subseteq \text{Co}(n)$ , then  $|H| < |H_n|$ .

(Note that for  $n = 13$ , the additional requirement that  $H \not\subseteq K_6 \cup K_6 \cup K_1$  is, due to connectedness, fulfilled automatically.)

We devote the last two subsections of Section 4 to prove Lemma 5. Meanwhile, taking Lemma 5 for granted, let us quickly complete the proof of Theorem 9. Assume that  $H$  is *disconnected* and, as before, let  $m = m(H)$  be the order of the smallest component of  $H$ ,  $1 \leq m \leq n - m$ ,  $m \neq 2$ . Below, to bound  $|H|$ , we use the Turán numbers for  $P$  of the 1st, 2nd, and 3rd order and utilize, respectively, Theorems 6 and 7, and 9 (per induction).

If  $v$  is an isolated vertex ( $\mathbf{m} = \mathbf{1}$ ), then, similarly as for small  $n$ ,  $H - v$  satisfies (3), because otherwise  $H$  would not. Hence, for  $n \geq 15$ ,

$$|H| \leq \text{ex}^{(3)}(n - 1; P) < |H_n|.$$

For  $n = 14$ , we cannot guarantee that  $H - v \not\subseteq K_6 \cup K_6 \cup K_1$ , so we use the second order Turán number instead which still does the job:

$$|H| \leq \text{ex}^{(2)}(13; P) = 40 < 41 = |H_{14}|.$$

To complete the case  $m = 1$  notice that for  $n = 13$ , since  $H \not\subseteq K_6 \cup K_6 \cup K_1$ , we have  $H - v \not\subseteq K_6 \cup K_6$ , and we are in position to use induction again. Hence,

$$|H| \leq \text{ex}^{(3)}(12; P) = 32 < 35 = |H_{13}|.$$

For  $m \geq 3$ , let us express  $H$  as a vertex disjoint union of two 3-graphs:

$$H = H' \cup H'', \quad |V(H')| = m, \quad |V(H'')| = n - m.$$

Then, clearly, both  $H'$  and  $H''$  are  $P$ -free, and thus

$$|H| \leq \text{ex}^{(1)}(m; P) + \text{ex}^{(1)}(n - m; P). \tag{4}$$

For  $\mathbf{m} = \mathbf{3}$ , since  $H \not\subseteq \text{Co}(n)$ , we have  $H'' \not\subseteq S_{n-3}$  and consequently

$$|H| \leq 1 + \text{ex}^{(2)}(n - 3; P) < |H_n|,$$

where the last inequality is easily verified by hand.

For  $\mathbf{m} = \mathbf{4}$  and  $n \in \{13, 14\}$  by (4),

$$|H| \leq 4 + \text{ex}^{(1)}(n - 4; P) = 4 + \binom{n - 5}{2} < 20 + \binom{n - 7}{2} = |H_n|;$$

and for  $n \geq 15$ , either  $H'' \subseteq S_{n-4}$  and so  $H \subseteq K_4 \cup S_{n-4} = H_n$  (in which case we are done) or  $H'' \not\subseteq S_{n-4}$  but then,

$$|H| \leq 4 + \text{ex}^{(2)}(n - 4; P) < 4 + \binom{n - 5}{2} = |H_n|.$$

For  $\mathbf{m} = \mathbf{5}$  by (4),

$$|H| \leq \text{ex}^{(1)}(5; P) + \text{ex}^{(1)}(n - 5; P) = 10 + \binom{n - 6}{2} < |H_n|.$$

For  $\mathbf{m} = \mathbf{6}$  and  $n = 13$ , since  $H \not\subseteq K_6 \cup K_6 \cup K_1$  we have  $H'' \not\subseteq K_6 \cup K_1$  and so,  $|H''| \leq \text{ex}^{(2)}(7; P) = |S_7|$  whereas for  $n \geq 14$ , we bound  $|H''| \leq \text{ex}^{(1)}(n - 6; P) = |S_{n-6}|$ . Hence, in both cases we have

$$|H| \leq \text{ex}^{(1)}(6; P) + |S_{n-6}| = 20 + \binom{n - 7}{2} \leq |H_n|.$$

However, for  $n \in \{13, 14\}$ , the first inequality must be strict (since  $H \neq H_n$ ), while for  $n \geq 15$  the second inequality is strict.

For  $\mathbf{m} = \mathbf{7}$  we have  $n \geq 14$  and, by (4), for  $n = 14$

$$|H| \leq \text{ex}^{(1)}(7; P) + \text{ex}^{(1)}(7; P) = 20 + 20 < 41 = |H_{14}|,$$

whereas, for  $n \geq 15$

$$|H| \leq \text{ex}^{(1)}(7; P) + \text{ex}^{(1)}(n - 7; P) = 20 + \binom{n - 8}{2} < 4 + \binom{n - 5}{2} = |H_n|.$$

Finally, for  $\mathbf{m} \geq \mathbf{8}$  we have  $n \geq 16$  and, by (4),

$$\begin{aligned} |H| &\leq \text{ex}^{(1)}(m; P) + \text{ex}^{(1)}(n - m; P) = \binom{m - 1}{2} + \binom{n - m - 1}{2} \leq \binom{7}{2} + \binom{n - 9}{2} \\ &< \binom{3}{2} + \binom{n - 5}{2} < 4 + \binom{n - 5}{2} = |H_n|. \end{aligned}$$

### 4.3. Preparations for the proof of Lemma 5

Under the assumptions on  $H$  in Lemma 5, let  $Q$  be a copy of  $P_2$  in  $H$  such that there is at least one edge disjoint from  $U = V(Q)$ . We know that  $Q$  exists, because  $P_2 \cup K_3 \subset H$ . Let  $V = V(H)$  and  $W = V \setminus U$ . Further, let  $W_0$  be the set of vertices of degree zero in  $H[W]$  and  $W_1 = W \setminus W_0$  (see Fig. 2). Note that, by definition,  $H[W] = H[W_1]$  and  $|W_1| \geq 3$ .



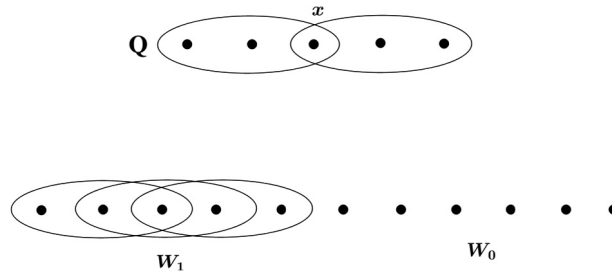


Fig. 2. Set-up for the proof of Lemma 5.

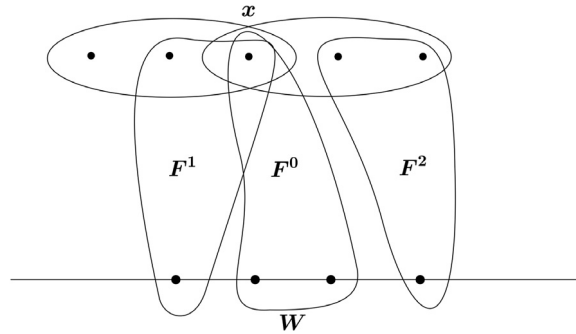


Fig. 3. Three types of edges in  $H_0 \cup H_1$ .

We also split the set of edges of  $H$ . First, notice that, since  $H$  is  $P$ -free, there is no edge with one vertex in each  $U$ ,  $W_0$ , and  $W_1$ . Let for  $i = 0, 1$ ,  $H_i$  be the sub-3-graph of  $H$  composed of the edges intersecting both  $U$  and  $W_i$ . Then, clearly,

$$H = H[U] \cup H[W] \cup H_0 \cup H_1, \tag{5}$$

with all four parts edge-disjoint.

Let  $x$  be the vertex of degree two in  $Q$ . If for some  $h \in H_0 \cup H_1$  we have  $|h \cap U| = 1$ , then  $h \cap U = \{x\}$ , since otherwise  $h$  together with  $Q$  would form a copy of  $P$  in  $H$ . We let

$$F^0 = \{h \in H_0 \cup H_1 : h \cap U = \{x\}\}.$$

Also, the edges  $h \in H_0 \cup H_1$  with  $|h \cap U| = 2$  must be such that the pair  $h \cap U$  is contained in an edge of  $Q$ , since otherwise  $h$  together with  $Q$  would form a copy of  $C$  in  $H$ . For  $k = 1, 2$ , define

$$F^k = \{h \in H_0 \cup H_1 : |h \cap U \setminus \{x\}| = k\}.$$

We have  $H_0 \cup H_1 = F^0 \cup F^1 \cup F^2$  (see Fig. 3). Further, for  $i = 0, 1$  and  $k = 0, 1, 2$ , we set

$$F_i^k = F^k \cap H_i.$$

Note that, since  $H$  is  $P$ -free,  $F_1^1 = \emptyset$  and thus,

$$H_1 = F_1^0 \cup F_1^2. \tag{6}$$

Observe also that, because  $H$  is connected,  $H_1 \neq \emptyset$ . Consequently, since the presence of any edge of  $H_1$  forbids at least 4 edges of  $H[U]$ ,

$$|H[U]| \leq 6. \tag{7}$$

In [11, Fact 3] the authors have proven the following bound on the number of edges in  $H_1$ :

$$|H_1| \leq 2|W_1| - 3. \tag{8}$$

We use (8) to estimate  $|H[W]| + |H_1|$ .



**Fact 1.** Set  $|W_1| = z$ . We have

$$|H[W]| + |H_1| \leq \begin{cases} \binom{z}{3} + 2z - 3 & \text{for } 3 \leq z \leq 5, \\ \binom{z-1}{2} + 2z - 3 & \text{for } 6 \leq z \leq 7, \\ 28 & \text{for } z = 8, \\ \frac{(z-1)^2}{2} + 2 & \text{for } z \geq 9. \end{cases} \tag{9}$$

**Proof.** For  $3 \leq z \leq 5$ , the above inequality is an immediate consequence of (8), whereas for  $6 \leq z \leq 7$ , we use (8) and the bound  $|H[W]| \leq \binom{z-1}{2}$ , stemming from Theorem 5.

For  $z \geq 8$  we have to consider two cases. Suppose first that  $H[W] \subseteq S_z$  with the center  $v \in W_1$ . Since  $H$  is  $P$ -free, every edge  $h \in F_1^2$  must have  $h \cap W_1 = \{v\}$ . Hence,  $|F_1^2| \leq 2$ . Moreover, if  $e \in F_1^0$ , then the pair  $e \cap W_1$  must be *nonseparable* in  $H[W_1]$ , that is, every edge of  $H[W_1]$  must contain both these vertices or none. Since, as it can be easily proved, there are at most  $\lfloor \frac{z-1}{2} \rfloor$  nonseparable pairs in  $W_1$ ,

$$|F_1^0| \leq \left\lfloor \frac{z-1}{2} \right\rfloor.$$

Consequently, by (6),

$$|H_1| \leq 2 + \left\lfloor \frac{z-1}{2} \right\rfloor$$

and, again using Theorem 5,

$$|H[W]| + |H_1| \leq \binom{z-1}{2} + 2 + \left\lfloor \frac{z-1}{2} \right\rfloor \leq \frac{(z-1)^2}{2} + 2. \tag{10}$$

Otherwise, that is, when  $H[W] \not\subseteq S_z$ , we consider two further subcases. If  $H[W] \not\supseteq M$ , then, by Theorem 3 combined with the fact that  $C \subset G_i(n)$ ,  $i = 1, 2$ , and the assumption that  $C \not\subseteq H$ ,

$$|H[W]| \leq 3z - 9. \tag{11}$$

If, on the other hand,  $H[W] \supseteq M$ , then, by Lemma 2,

$$|H[W]| \leq \text{ex}(n; \{P, C\} | M) = \begin{cases} 2z - 4 & \text{for } 8 \leq z \leq 9, \\ 20 & \text{for } z = 10, \\ 4 + \binom{z-4}{2} & \text{for } z \geq 11. \end{cases} \tag{12}$$

Taking the larger of the bounds (11) and (12), and using (8), we arrive at the ultimate bound

$$|H[W]| + |H_1| \leq \begin{cases} 5z - 12 & \text{for } 8 \leq z \leq 10, \\ \binom{z-4}{2} + 2z + 1 & \text{for } z \geq 11. \end{cases} \tag{13}$$

It is easy to check that for  $z \geq 8$ , the maximum of the R-H-S of (10) and the R-H-S (13) is equal to the R-H-S of (9).  $\square$

#### 4.4. Proof of Lemma 5

We adopt notation from the previous subsection. In particular, recall that  $z = |W_1|$ . Additionally, we set  $s = |W_0|$ . Our plan is to first give the proof in three ‘smallest’ cases:  $s = 0, z = 3$ , and  $n \in \{13, 14, 15\}$ .  $\mathbf{s = 0}$  ( $W_0 = \emptyset$ ). Then  $H_0 = \emptyset$  and  $z = n - 5 \geq 8$ . By (5), (7), and (9),

$$|H| = |H[U]| + |H[W]| + |H_1| \leq \begin{cases} 34 & \text{for } z = 8, \\ 6 + \frac{(z-1)^2}{2} + 2 & \text{for } z \geq 9. \end{cases}$$

This implies that for  $n = 13, |H| < 35$ , for  $n = 14, |H| \leq 40 < 41$ , while for  $n \geq 15$ , it can be easily checked that

$$|H| \leq 6 + \frac{(z-1)^2}{2} + 2 < 4 + \binom{z}{2} = |H_n|.$$

Therefore, from now on we will be assuming that  $W_0 \neq \emptyset$ , or  $s \geq 1$ . The proofs of the other two special cases,  $z = 3$  and  $n \in \{13, 14, 15\}$ , are both split into two subcases with respect to  $F_1^2$ . We begin with bounding the number of edges in  $H[U \cup W_0]$  when  $F_1^2 \neq \emptyset$ .

**Fact 2.** If  $F_1^2 \neq \emptyset$ , then

$$|H[U \cup W_0]| \leq \begin{cases} 8 & \text{for } s = 1, \\ 3s + 7 & \text{for } 2 \leq s \leq 4, \\ \binom{s+2}{2} + 1 & \text{for } s \geq 5. \end{cases} \tag{14}$$

**Proof.** Let  $u$  and  $v$  be some two vertices of  $U$  belonging to the same edge of  $F_1^2$ . If  $H[U \cup W_0] \subseteq S_{s+5}$  (with the center in  $x$ ), then, since  $H$  is  $P$ -free, the only edge of  $H[U \cup W_0]$  containing  $u$  or  $v$  is  $\{x, u, v\}$ . Hence

$$|H[U \cup W_0]| \leq \binom{s+2}{2} + 1. \tag{15}$$

Otherwise, either  $H[U \cup W_0] \not\supseteq M$  and, assuming that  $s \geq 2$ , and thus  $|U \cup W_0| \geq 7$ , by [Theorem 3](#),

$$|H[U \cup W_0]| \leq \text{ex}^{(2)}(s+5; M) = 3(s+5) - 8 = 3s + 7, \tag{16}$$

or  $H[U \cup W_0] \supseteq M$  and, by [Lemma 2](#), this time including  $s = 1$ ,

$$|H[U \cup W_0]| \leq \text{ex}(s+5; \{P, C\}|M) = \begin{cases} 2s + 6 & \text{for } 1 \leq s \leq 4, \\ 20 & \text{for } s = 5, \\ 4 + \binom{s+1}{2} & \text{for } s \geq 6. \end{cases} \tag{17}$$

For  $s = 1$  we argue as follows. Since  $H$  is  $P$ -free, every edge of  $H[U \cup W_0]$  must contain either both of  $u$  and  $v$  or none. There are only 8 such edges and so,  $|H[U \cup W_0]| \leq 8$ . In summary, (14) follows by (15), (16), (17), and the above bound for  $s = 1$ .  $\square$

$z = 3, F_1^2 \neq \emptyset$ . We combine bounds (9) of [Fact 1](#) and (14) to estimate  $|H|$ . Since  $s = n - 5 - z \geq 13 - 8 = 5$ ,

$$\begin{aligned} |H| &= |H[U \cup W_0]| + |H_1| + |H[W]| \leq \binom{s+2}{2} + 1 + \binom{3}{3} + 2 \cdot 3 - 3 \\ &= \binom{n-6}{2} + 5 < |H_n|. \end{aligned}$$

$z = 3, F_1^2 = \emptyset$ . Then, by (6),  $|H_1| \leq |F_1^0| \leq 3$ . Since  $H \not\subseteq \text{Co}(n)$  we have  $H[U \cup W_0] \not\subseteq S_{s+5}$  and consequently, by [Theorem 7](#),

$$|H[U \cup W_0]| \leq \text{ex}^{(2)}(s+5; P) = \begin{cases} 20 + \binom{s-1}{3} & \text{for } 10 \leq s+5 \leq 12, \\ 4 + \binom{s+1}{2} & \text{for } s+5 \geq 13. \end{cases}$$

Hence, for  $13 \leq n \leq 15$  ( $10 \leq s+5 \leq 12$ )

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 20 + \binom{s-1}{3} + 3 + 1 = 24 + \binom{n-9}{3} < |H_n|,$$

while for  $n \geq 16$  ( $s+5 \geq 13$ )

$$\begin{aligned} |H| &= |H[U \cup W_0]| + |H_1| + |H[W]| \leq 4 + \binom{s+1}{2} + 3 + 1 \\ &= 8 + \binom{n-7}{2} < 4 + \binom{n-5}{2} = |H_n|. \end{aligned}$$

Consequently for the rest of the proof we will be assuming that  $z \geq 4$  (and  $s \geq 1$ ).

$n \in \{13, 14, 15\}, F_1^2 \neq \emptyset$ . We again combine bounds (9) of [Fact 1](#) and (14) to estimate  $|H|$ . For  $n = 13 = 5 + s + z$ , where  $4 \leq z \leq 7$ , the worst case is when  $z = 7$  and  $s = 1$ , in which we get

$$|H| \leq 34 < 35.$$

For  $n = 14 = 5 + s + z$ , where  $4 \leq z \leq 8$ , the worst case is when  $z = 7$  and  $s = 2$ , and so

$$|H| \leq 39 < 41.$$

For  $n = 15 = 5 + s + z$ , where  $4 \leq z \leq 9$ ,

$$|H| \leq 42 < 49.$$

$n \in \{13, 14, 15\}$ ,  $F_1^2 = \emptyset$ . Since  $H$  is  $C$ -free, by Theorem 5,

$$|H[U \cup W_0]| \leq \binom{s+4}{2}. \tag{18}$$

For  $z = 4$ , by an easy inspection, one can show that  $|H_1| + |H[W]| \leq 1 + 2 = 3$ . Therefore, by (18), for  $n = 13$ ,  $|H| \leq 28 + 3 < 35$ , for  $n = 14$ ,  $|H| \leq 36 + 3 < 41$ , and for  $n = 15$ ,  $|H| \leq 45 + 3 < 49$ .

For  $z \geq 5$ , we may apply Theorem 5 also to  $H[W_1 \cup \{x\}]$ , obtaining the bound

$$|H_1| + |H[W]| = |H[W_1 \cup \{x\}]| \leq \binom{z}{2}.$$

In summary, by (18),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq \binom{s+4}{2} + \binom{z}{2}.$$

Consequently, by choosing optimal pairs  $(z, s)$ , for  $n = 13 = 5 + s + z$ , where  $5 \leq z \leq 7$ , we get

$$|H| \leq 31 < 35,$$

for  $n = 14 = 5 + s + z$ , where  $5 \leq z \leq 8$ , we get

$$|H| \leq 38 < 41,$$

whereas for  $n = 15 = 5 + s + z$ , where  $5 \leq z \leq 9$ ,

$$|H| \leq 46 < 49.$$

Thus, we are done with the proof of Lemma 5 in all three cases:  $s = 0, z = 3$ , and  $n \in \{13, 14, 15\}$ . In fact, recalling our argument from Section 4.2, we have actually proved Theorem 9 for all  $n \leq 15$ . To complete the proof of Lemma 5 for the remaining values of  $n$ , we need only to prove Fact 3 below. The proof is by induction on  $n$ , and we include the case  $n = 15$  there to serve as the inductive base.

Note that compared to Lemma 5, we now relax the assumption of connectivity, replacing it with that of  $H_1 \neq \emptyset$ . Also, although we have already proved Lemma 5 for  $s = 0$ , or  $W_0 = \emptyset$ , we do not impose the opposite assumption here. Both these relaxations are made to accommodate the inductive proof below. Finally, note that we may drop the assumption that  $H \not\subseteq \text{Co}(n)$ , as it follows from the fact that  $|W_1| \geq 4$  (a comet cannot contain two edges not containing the center). For a 3-graph  $G$  and a vertex  $v \in V(G)$ , let  $G(v)$  denote the link graph of  $v$  in  $G$ , that is, the set of pairs of vertices which together with  $v$  form an edge of  $G$ .

**Fact 3.** For  $n \geq 15$ , if  $H$  is an  $n$ -vertex,  $\{P, C\}$ -free 3-graph such that  $H \supset P_2 \cup K_3$  and, under the notation of Section 4.3,  $z = |W_1| \geq 4$  and  $H_1 \neq \emptyset$ , then

$$|H| < 4 + \binom{n-5}{2} = |H_n|.$$

**Proof.** The proof is by induction on  $n$  with the initial step  $n = 15$  done earlier. Let  $n \geq 16$ . It can be easily checked that, since  $H$  is  $P$ -free, for every  $v \in W$  either

$$F^0(v) = \emptyset \quad \text{or} \quad F^2(v) = \emptyset. \tag{19}$$

Moreover, by the definitions of  $F^1$  and  $F^2$ ,

$$|F^1(v)| \leq 4 \quad \text{and} \quad |F^2(v)| \leq 2. \tag{20}$$

If  $W_0 = \emptyset$ , then we are done by an earlier proof (at the beginning of this subsection). Otherwise, fix  $v \in W_0$  and observe that, by the remark preceding (5),  $|F^0(v)| \leq |W_0| - 1$ . Thus, by (19) and (20), since  $|W_0| = n - 5 - |W_1| \leq n - 9$ ,

$$|H(v)| = |F^0(v)| + |F^1(v)| + |F^2(v)| \leq 4 + \max\{2, |W_0| - 1\} \leq 4 + n - 10 = n - 6. \tag{21}$$

Notice that  $H - v$  satisfies the assumptions of Fact 3. Indeed, as the removal of  $v$  affects  $H_0$  only, in the obtained sub-3-graph we still have both,  $H_1 \neq \emptyset$  and  $|W_1| \geq 4$ . Consequently, by the induction's assumption and (21)

$$|H| = |H - v| + |H(v)| < 4 + \binom{n-6}{2} + n - 6 = 4 + \binom{n-5}{2} = |H_n|. \quad \square$$

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