Rachunek Prawdopodobieństwa 2

Zestaw zadań nr 6

Termin realizacji: 19 XII 2008

1. Let $l_{ij}(n) = P(X_n = j, X_m \neq i, 1 \leq m \leq n | X_0 = i)$ be the probability that the chain passes from i to j in n steps without revisiting i (the last exit time). Writing, $L_{ij}(s) = \sum_{n=1}^{\infty} s^n l_{ij}(n)$, show that $P_{ij}(s) = P_{ii}(s)L_{ij}(s)$ if $i \neq j$. Deduce that the first passage times and the last exit times have the same distribution for any Markov chain for which $P_{ii}(s) = P_{jj}(s)$ for all i and j. Give an example of such a chain.

2. The distinct pair i, j of states of a Markov chain is called *symmetric* if $P(T_j < T_i | X_0 = i) = P(T_i < T_j | X_0 = j)$. If $X_0 = i$ and i, j is symmetric, find the expected number of visits to j before the chain revisits i.

3. A particle performs a random walk on the vertices of the cube. At each step it remains where it is with probability 1/4, or moves to one of its neighboring vertices each having probability 1/4. Let v and w be two diametrically opposite vertices. If the walk starts from v, find: (a) the mean number of steps until its first return to v, (b) the mean number of steps until its first visit to w, (c) the mean number of visits to w before its first return to v.

4. Show that if a Markov chain is irreducible and has a state i such that $P_{ii} > 0$, then it is also aperiodic.

5. Consider a chessboard with a lone white king making (legal) moves uniformly at random. Is the corresponding Markov chain irreducible and aperiodic? Same question for a bishop and a knight. In the last case, show that if the chain starts from a fixed square (say, a1), then $\mu^{(n)}$ does not converge in total variation.

6. Skonstruuj redukowalny łańcuch Markowa o 4 stanach, ktory posiada 2 różne rozkłady stacjonarne. Wskaż te rozkłady.

7. Pokazać, że d_{TV} jest metryką oraz, że spełnia wzór ze strony 6-9 (przed definicją 9)

8. Pokazać, że $P(X = Y) \leq 1 - d_{TV}(X, Y)$.

9. X i Y mają rozkład Poissona z parametrami λ i μ, odpowiednio. Jeśli $\lambda \geq \mu$ to X stochastycznie dominuje Y.

10. Let X and Y be indicator random variables with $EX = p$ and $EY = q$. What is the maximum value of $P(X = Y)$ as a function of p and q. Explain how X, Y need to be distributed in order that $P(X = Y)$ be (a) maximized, (b) minimized.