Rachunek Prawdopodobieństwa 2

Zestaw zadań nr 7

Termin realizacji: 19 XII 2008

1. Can a reversible chain be periodic?
2. (The Ehrenfest model of diffusion, a.k.a. "dog-flea model") Consider a Markov chain with transition probabilities $p_{i,i+1} = 1 - i/m$, $p_{i,i-1} = i/m$, for $0 \leq m \leq m$. Show that if $X_0 = i$,

$$E(X_n - m/2) = (i - m/2)(1 - 2/m)^n \to 0,$$

   as $n \to \infty$ and $m \geq 2$. Also, find a stationary distribution.

3. (Dla niedyskretnych) The Markov-Kakutani theorem asserts that , for any convex compact subset C of R^n and any linear continuous mapping T of C into C, T has a fixed point.

   Let T be a $m \times n$ matrix and let $v \in R^n$. Farkas's theorem asserts that exactly one of the following holds:

   (i) there exists $x \in R^n$ such that $x \geq 0$ and $xT = v$

   (ii) there exists $y \in R^n$ such thast $yv^T < 0$ and $Ty^T \geq 0$

   Use either of the above theorems to prove that a finite stochastic matrix has a non-negative non-zero left eigenvector corresponding to the eigenvalue 1. Conclude that a Markov chain with finitely many states has a stationary distribution.

4. Consider a chessboard with a lone white king making (legal) moves uniformly at random. What is the mean recurrence time of a corner square? Same question for queen, bishop, knight, rook. (Hint: find the stationary distribution first)
5. A rook and a bishop perform independent symmetric random walks with synchronous steps on a 4×4 chessboard. If they start from a corner, find the expected number of steps until they meet again at the same corner.
6. Consider a symmetric random walk on a 3-dimensional integer lattice, starting at $(0, 0, 0)$. Find an exact formula for $P(X_{2n} = (0, 0, 0))$ and deduce, by Stirling's formula, that the ultimate return to the origin is not certain.
7. Consider the reducible Markov chain given by the transition matrix

$$P = \begin{vmatrix} .5 & .3 & 0 & 0 \\ .3 & .7 & 0 & 0 \\ 0 & 0 & .2 & .8 \\ 0 & 0 & .8 & .2 \end{vmatrix},$$

   Find two different stationary distributions.