SOLITARY SUBGRAPHS OF RANDOM GRAPHS

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ABSTRACT. Let G be a graph and let $\mathcal{G}(n,p)$ be the binomial random graph with n vertices and edge probability p. We consider copies of G in $\mathcal{G}(n,p)$, vertex disjoint from all other such copies. For a strictly balanced graph G, initially, every copy of G in $\mathcal{G}(n,p)$ is solitary. Suen [4] established a second (disappearence) threshold for a subclass of strictly balanced graphs. In this paper we extend his result to a more general case.

1. Introduction

A random graph $\mathcal{G}(n,p)$ is a graph obtained from the complete graph K_n by independent deletion of each edge with probability $1-p$. We say that a random graph possesses a property Q *asymptotically almost surely (aas)* if the probability that this random graph possesses Q converges to 1 as $n \to \infty$. In this paper we will often use, for convenience, notation $a_n \asymp b_n$ instead of $a_n = \Theta(b_n)$. For a graph G, let v_G and l_G stand for its number of vertices and edges, respectively. If a subgraph H of a graph F is isomorphic to a graph G, then H is called a copy of G in F.

Fix a graph G and denote by $G_1, G_2, \ldots, G_t$, $\quad t = \binom{n}{v_G}\frac{v_G!}{aut(G)}$, all copies of G in the complete graph K_n, where $aut(G)$ stands for the number of automorphisms of the graph G. For each $i = 1, 2, \ldots, t$ define the indicator random variable

$$I_i = I_i(n,p) = \begin{cases} 1 & \text{if} \quad G_i \subset \mathcal{G}(n,p) \\ 0 & \text{otherwise.} \end{cases}$$

Then $X = X_G = X_G(n,p) = \sum_{i=1}^{t} I_i$ counts the subgraphs of a random graph $\mathcal{G}(n,p)$ isomorphic to G.

Define the density of G as $d_G = \frac{l_G}{v_G}$, $v_G \geq 1$, and let $m_G = \max_{H \subseteq G} d_H$. A graph G is *balanced* if $m_G = d_G$, and *strictly balanced* if for every $H \subset G, d_H < d_G$.

In 1981 Bollobás [1] proved the following result.

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Theorem 1.

$$\lim_{n\to\infty} P(X_G > 0) = \begin{cases} 0 & \text{if} \quad np^{m_G} \to 0 \\ 1 & \text{if} \quad np^{m_G} \to \infty. \end{cases}$$

Moreover, if G is a strictly balanced graph and $np^{m_G} \to c$ as $n \to \infty$ then X_G converges to the Poisson distribution with expectation $\frac{c^{v_G}}{aut(G)}$.

The threshold part of the above theorem was proved for balanced graphs already by Erdős and Rényi in 1960 [2]. The distribution part was shown, independently from Bollobás, by Karoński and Ruciński [3].

Let $Z = Z_G(n,p)$ be the number of copies of G in $\mathcal{G}(n,p)$ which are vertex disjoint from all other copies of G in $\mathcal{G}(n,p)$. We shall call such copies *solitary.* If G is strictly balanced and $p = \Theta(n^{-\frac{1}{m_G}})$, then *aas* there are no intersecting pairs of G at all, and so $Z = X$. Indeed, if Y denotes the number of pairs of distinct copies of G which intersect each other then

$$P(Y > 0) \le EY \asymp \sum_{H \subset G} n^{2v_G - v_H} p^{2l_G - l_H} = o(1) \ ,$$

since $n^{v_H} p^{l_H} = (np^{d_H})^{v_H} \to \infty$ for all $H \subset G$.

In the next section we will prove the following preliminary result, which exhibits the special role of striclty balanced graphs in the context of solitary subgraphs of $\mathcal{G}(n,p)$.

Proposition.

1. *If G is not balanced then for every $p = p(n)$, $P(Z_G > 0) = o(1)$;*
2. *If G is balanced but not strictly then for every $p = p(n)$ such that $p = o(n^{-\frac{1}{m_G}})$ or $p \gg n^{-\frac{1}{m_G}}$ we have $P(Z_G > 0) = o(1)$;*
3. *If G is balanced but not strictly and $p = \Theta(n^{-\frac{1}{m_G}})$ then*

$$0 < \liminf_{n\to\infty} P(Z_G > 0) \le \limsup_{n\to\infty} P(Z_G > 0) < 1.$$

Our main question is for what range of $p = p(n)$, $P(Z_G > 0) \to 1$? In view of the Proposition it only makes sense to raise this question for strictly balanced graphs. Indeed, if for some $p = p(n)$ there *aas* exist solitary copies of G then G must necessarily be strictly balanced. This question was answered by Suen [4] in the special case when G is *strictly strongly balanced* , i.e. for every $H \subset G$, $\frac{l_H}{v_H - 1} < \frac{l_G}{v_G - 1}$ (see Theorem 3 below).

In order to formulate our result we need to introduce a few more definitions. Given a sequence $p = p(n)$, we call a subgraph H of G *a leading overlap* of G if $EX_H = O(EX_K)$ for all $K \subseteq G$. Clearly each leading overlap is an induced subgraph of G. Moreover it can be easily verified that when $np^{m_G} \to \infty$, each leading overlap must be a connected subgraph of G. As

$$EX_G = \sum_{i=1}^{t} EI_i = \binom{n}{v_G} \frac{v_G!}{aut(G)} p^{l_G} \asymp n^{v_G} p^{l_G} \ ,$$

and, similarly, $EX_H \asymp n^{v_H} p^{l_H}$, for every subgraph H of G, the densities $d_H = \frac{l_H}{v_H}$ of all subgraphs of G play a decisive role in the determination of leading overlaps of G.

The *subgraph plot* of a graph G is defined as the set of points

$$\Gamma(G) = \{(v_H, l_H) : \quad H \subseteq G\}.$$

The upper boundary of the convex hull of $\Gamma(G)$ will be called here *the roof.*

Observe that a subgraph H is, for some range of $p = p(n)$, a leading overlap of G if and only if it lies on the roof. Moreover, the slopes of the line segments to the left and to the right of H determine the range of p for which H is a leading overlap during the evolution of $\mathcal{G}(n,p)$. Let *the spectrum* of G be defined as the collection $Spec(G)$ of the leading overlaps of G ordered by decreasing number of vertices. Equivalently, $Spec(G)$ is formed by the subgraphs of G plotted on the roof and ordered from right to left (if two of them are plotted at the same point, their order is immaterial). The first element of the spectrum is always G itself, and the last one is always K_1. Observe that if G, H_1 and H_2 are three initial elements of $Spec(G)$ lying on the same line segment of the roof (including the case when the points coincide), then H_1 and H_2 become leading overlaps at the same moment of the evolution of $\mathcal{G}(n,p)$, precisely when $p \asymp n^{-\frac{l_G - l_{H_1}}{v_G - v_{H_1}}} = n^{-\frac{l_G - l_{H_2}}{v_G - v_{H_2}}}$ (in fact, as soon as $pn^{\frac{l_G - l_{H_1}}{v_G - v_{H_1}}} \to \infty$, H_1 drops off momentarily). On the other hand, if $Spec(G) = (G, H, ..., K_1)$ and no other subgraph of G is plotted on the straight line passing through G and H, then H will be referred to as *the unique second leader* .

Fact. *If H is the unique second leader of G then any two copies of H in G must be disjoint.*

Proof. Let for every $F \subseteq G, \quad F \neq \emptyset, \quad f(F) = \alpha(v_G - v_F) - (l_G - l_F)$. This function is modular, i.e. $f(G_1 \cup G_2) = f(G_1) + f(G_2) - f(G_1 \cap G_2)$. For $F \neq H$, and $F \neq G \quad f(F) < 0$, moreover $f(H) = f(G) = 0$. Let F be a union of two copies of H which intersect on K $(K \neq \emptyset)$. Then $f(F) = 2f(H) - f(K) = -f(K)$, which is a contradition, unless $K = H = F$. □

Let us consider a few examples.

Example 1.

$G =$

$Spec(G) = (G, K_3, K_1)$.

Fig.1

Here K_3 is the unique second leader

If G is strictly strongly balanced then always K_1 is the unique second leader of G.

Example 2.

$G =$

$Spec(G) = (G, K_1)$.

Fig.2

The main theorem of this paper concerns the subclass of strictly balanced graphs G with the unique second leader. The next two examples show graphs for which there are no unique second leader and thus Theorem 2 does not apply to them.

Example 3.

$G =$

$Spec(G) = (G, T_6, T_6', T_5, T_5', T_4, T_4', T_3, K_2, K_1)$,

Fig.3

where the elements of the spectrum are all subgraphs of G being trees (the subscripts represent the number of vertices of a tree; where there are more than one tree on the same number of vertices, the superscripts are used).

Example 4.

$G =$

$Spec(G) = (G, K_4, K_1)$.

Fig.4

In the next section we will prove our main result. For a graph G and its subgraph H, let $f(H, G)$ be the number of copies of H in G.

Theorem 2.
Let G be a strictly balanced graph, H be the unique second leader of G and let

$$\omega(n) = n^{v_G - v_H} p^{l_G - l_H} - \frac{\alpha v_G - l_G}{\alpha s} \log n - \frac{l_G}{s(l_G - l_H)} \log\log n,$$

where $s = \frac{[f(H,G)]^2 aut(H)}{aut(G)}$ *and* $\alpha = \frac{l_G - l_H}{v_G - v_H}$. *Then*

$$\lim_{n\to\infty} P(Z_G > 0) = \begin{cases} 0 & \text{if} \quad \omega(n) \to \infty \quad \text{or} \quad np^{d_G} \to 0 \\ 1 & \text{if} \quad \omega(n) \to -\infty \quad \text{and} \quad np^{d_G} \to \infty. \end{cases}$$

Moreover, if $np^{d_G} \to c$ *or* $\omega(n) \to c$, *then* Z_G *converges to the Poisson distribution with expectation, respectively,* $\frac{c^{v_G}}{aut(G)}$ *or* $\frac{1}{aut(G)} \left[\frac{\alpha v_G - l_G}{\alpha s}\right]^{\frac{l_G}{l_G - l_H}} e^{-sc}$.

Thus, in addition to the known threshold $n^{-\frac{1}{d_G}}$ for the appearance of solitary copies of G in $\mathcal{G}(n,p)$, Theorem 2 establishes a second (disappearance) threshold around $(\log n)^{\frac{1}{l_G - l_H}} n^{-\frac{1}{\alpha}}$.

Because for a strictly strongly balanced graph, K_1 is always the unique second leader, one can easily see that the following theorem of Suen, which was mentioned before, is an immediate consequence of Theorem 2.

Theorem 3 [Suen, 1990].
Let G be a strictly strongly balanced graph and

$$\omega(n) = n^{v_G - 1} p^{l_G} - {v_G}^{-2} aut(G) \log n - {v_G}^{-2} aut(G) \log\log n.$$

Then

$$\lim_{n\to\infty} P(Z_G > 0) = \begin{cases} 0 & \text{if} \quad \omega(n) \to \infty \quad \text{or} \quad np^{d_G} \to 0 \\ 1 & \text{if} \quad \omega(n) \to -\infty \quad \text{and} \quad np^{d_G} \to \infty. \end{cases}$$

Moreover, if $np^{d_G} \to c$ *or* $\omega(n) \to c$, *then* Z_G *converges to the Poisson distribution with expectation, respectively,* $\frac{c^{v_G}}{aut(G)}$ *or* $v_G^{-2} \exp\{-\frac{c {v_G}^2}{aut(G)}\}$.

Although we cannot apply Theorem 2 to graphs like those in Examples 3 and 4, we can find the second threshold in case when G is a tree. It is not hard to see that for the trees on 2 and 3 vertices as well as for the path on 4 vertices the notion of a solitary copy coincides with that of an isolated copy, thus trivially, the disappearence threshold for solitary copies is the same as one for isolated copies. In fact, the latter statement remains true for all trees.

Theorem 4.
Let G be a tree and let $\omega(n) = v_G np - \log n - (v_G - 1) \log\log n$. *Then*

$$\lim_{n\to\infty} P(Z_G > 0) = \begin{cases} 0 & \text{if} \quad \omega(n) \to \infty \quad \text{or} \quad np^{d_G} \to 0 \\ 1 & \text{if} \quad \omega(n) \to -\infty \quad \text{and} \quad np^{d_G} \to \infty. \end{cases}$$

Moreover, if $np^{d_G} \to c$ *or* $\omega(n) \to c$, *then* Z_G *converges to the Poisson distribution with expectation, respectively,* $\frac{c^{v_G}}{aut(G)}$ *or* $\left[v_G^{v_G - 1} aut(G) e^c\right]^{-1}$.

Exactly the same threshold has been established for isolated trees by Erdős and Rényi [2].

2. Proofs

Let $\mathcal{G}^*(n,p)$ be a random graph in which the edges of a fixed copy G_0 of G are present with probability 1 and the remaining edges are present with probability p, independently from each other. Let S be the number of copies of G in $\mathcal{G}^*(n,p)$ which are not vertex disjoint from G_0. Then

$$EZ = EXP(S=0) \ , \tag{1.1}$$

where X and Z were defined in the Introduction.

For a subgraph H of G, let S_H be the number of copies of G in $\mathcal{G}^*(n,p)$ which intersect G_0 on a subgraph isomorphic to H. Clearly $P(S=0) \le P(S_H=0)$ and from (1.1), by the first moment method,

$$P(Z>0) \le EXP(S_H=0). \tag{1.2}$$

For every copy G_i of G such that $G_i \cap G_0 \cong H$, where the symbol ' $\cong$ ' designates the relation of isomorphism, define a zero-one random variable J_i by

$$J_i = \begin{cases} 1 & \text{if} \quad G_i \subset \mathcal{G}^*(n,p) \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Then $S_H = \sum\limits_{i:G_i \cap G_0 \cong H} J_i$. We can estimate $P(S_H=0)$ in (1.2) by applying the inequality ([3])

$$P(S_H = 0) \le \exp\Big\{ - \frac{(ES_H)^2}{\Delta} \Big\}, \tag{1.4}$$

where $\Delta = \sum\sum E(J_iJ_j)$ and the summation extends over all pairs (i,j) for which $G_i \cap G_0 \cong H$, $G_j \cap G_0 \cong H$ and $(E(G_i) \cap E(G_j)) \setminus E(G_0) \neq \emptyset$.

Denote $\Psi_G = n^{v_G} p^{l_G}$ and observe that

$$\begin{aligned} &ES_H = \sum_{i:G_i \cap G_0 \cong H} EJ_i \asymp n^{v_G - v_H} p^{l_G - l_H} = \frac{\Psi_G}{\Psi_H} \ , \\ &\text{and} \\ &\Delta \asymp \sum_K \frac{{\Psi_G}^2}{\Psi_H \Psi_K} \ , \end{aligned} \tag{1.5}$$

where K runs over all $H \subset K \subseteq G$, $E(K) \setminus E(H) \neq \emptyset$.

2.1 Proof of Proposition.

Let H be a largest subgraph of a not strictly balanced graph G for which $d_H = m_G$ and $H \neq G$. We have $d_H \ge d_G$ and $d_H > d_G$ when G is nonbalanced. Moreover, for arbitrary K such that $G \supset K \supset H$, the inequality $d_K < d_H$ holds.

By (1.4) and (1.5)

$$P(S_H = 0) \le \exp\Big\{ - \frac{\Theta(1)}{\Psi_H \sum\limits_K \frac{1}{\Psi_K}} \Big\} \le \exp\Big\{ - \Theta\left(\frac{\Psi_0}{\Psi_H}\right) \Big\} \ , \tag{1.6}$$

where $\Psi_0 = \Psi_{H_0} = \min_K \Psi_K$ and the range of K is as in (1.5). Observe that, because H is an induced subgraph of G, for every K in that range $v_K > v_H$. In particular, $v_{H_0} > v_H$.

Moreover, if G is balanced, then either $d_{H_0} < d_H = d_G$, or $H_0 = G$ and $d_{H_0} = d_H = d_G$. If G is nonbalanced, then always $d_{H_0} < d_H$. Thus, from (1.2) and (1.6),

$$P(Z>0) \le \Theta(\Psi_G) \exp\Big\{ -\Theta\left(\frac{\Psi_0}{\Psi_H}\right)\Big\} =$$
$$= \Theta(\Psi_G) \exp\Big\{ -\Omega\left(\frac{(np^{d_{H_0}})^{v_{H_0}}}{(np^{d_H})^{v_H}}\right)\Big\} \le$$
$$\le \begin{cases} \Theta(\Psi_G)\exp\Big\{ -\Omega\left((np^{d_G})^{v_{H_0}-v_H}\right)\Big\} & \text{if} \quad d_{H_0} = d_H \\ \Theta(\Psi_G)\exp\Big\{ -\Omega\left((np^{d_{H_0}})^{v_{H_0}-v_H}\right)\Big\} & \text{if} \quad d_{H_0} < d_H \end{cases}$$
$$= \begin{cases} o(1) & \text{for } p = o(n^{-\frac{1}{m_G}}) \text{ or } p \gg n^{-\frac{1}{m_G}} \\ o(1) & \text{for every } p = p(n)\ . \end{cases}$$

This proves parts (1) and (2) of the Proposition.

If G is balanced but not strictly and $p \sim cn^{-\frac{1}{m_G}}$, $c > 0$, then $\Psi_G = \Theta(1)$ and the right hand side of (3) follows from (1.6). The left hand side follows trivially from Theorem 1, since $P(Z=0) \sim P(X=0) \to e^{-\frac{c^{v_G}}{aut(G)}} < 1$. □

2.2. Proof of Theorem 2.

Throughout this subsection, let H be the unique second leader of a strictly balanced graph G. Recall that both H and G are connected graphs and that every two copies of H in G must be vertex disjoint (cf. Fact).

Theorem 2 consists of three statements, which will be referred to, according to the limit, as 0-statement, 1-statement and Poisson-statement, respectively.

We show the 0-statement using the first moment method. Clearly if $np^{d_G} \to 0$ then $P(Z>0) \le P(X>0) = o(1)$. Assume that

$$p = n^{-\frac{1}{\alpha}}\left[\frac{\alpha v_G - l_G}{\alpha s}\log n + \frac{l_G}{s(l_G - l_H)}\log\log n + \omega(n)\right]^{\frac{1}{l_G - l_H}},$$

where $\omega(n) \to \infty$. From (1.2) we have

$$P(Z>0) \le EXP(S_H = 0) \le n^{v_G} P(S_H = 0). \tag{1.7}$$

The inequality (1.4) can be rewritten as

$$P(S_H = 0) \le \exp\Big\{ -\frac{(ES_H)^2}{ES_H + \Delta'}\Big\} = \exp\Big\{ -\frac{ES_H}{1 + ES_H\frac{\Delta'}{(ES_H)^2}}\Big\},$$

where $\Delta' = \sum\sum E(J_iJ_j)$ is the partial sum of Δ taken over all pairs (i,j), $i \ne j$.

Observe that for every K such that $H \subset K \ne G$

$$\frac{\Psi_K}{\Psi_H} = n^{v_K - v_H} p^{l_{\tilde H} - l_H} \ge n^{v_K - v_H - \frac{l_K - l_H}{\alpha}} (\log n)^{\frac{l_K - l_H}{l_G - l_H}} \ge n^{\varepsilon_K}\ ,$$

where $\varepsilon_K = \frac{1}{2}[v_{\tilde{H}} - v_H - \frac{l_K - l_H}{\alpha}]$.

Thus, by (1.5)

$$\frac{\Delta'}{(ES_H)^2} \asymp \sum_{\tilde{H}} \frac{\Psi_H}{\Psi_K} \leq \sum_K n^{-\varepsilon_K} < n^{-\varepsilon},$$

where $\varepsilon = \frac{1}{2} \min_K \varepsilon_K$. Hence

$$P(S_H = 0) \leq \exp\Big\{ - \frac{ES_H}{1 + n^{-\varepsilon} ES_H} \Big\}. \tag{1.8}$$

Consider now three cases with respect to the order of magnitude of ES_H.

Assume first that $ES_H \geq n^{\frac{2\varepsilon}{3}}$. Then $P(S_H = 0) \leq \exp\{-n^{\frac{\varepsilon}{2}}\}$, and by (1.10) $P(Z > 0) < n^{v_G} \exp\{-n^{\frac{\varepsilon}{2}}\} = o(1)$.

If $\log^2 n \leq ES_H < n^{\frac{2\varepsilon}{3}}$, then

$$P(S_H = 0) \leq \exp\{-ES_H(1 - n^{-\frac{\varepsilon}{3}})\} \leq \exp\Big\{ - \frac{ES_H}{2} \Big\},$$

and

$$P(Z > 0) \leq n^{v_G} \exp\{-\frac{\log^2 n}{2}\} = o(1) \ .$$

Finally, if $ES_H \leq \log^2 n$ then $P(S_H = 0) \leq \exp\{-ES_H + o(1)\}$ and

$$P(Z_G > 0) \leq EX_G \exp\{-ES_H + o(1)\}. \tag{1.9}$$

Let us determine an asymptotic formula for ES_H. Let $c(H, G)$ stand for the number of graphs on the vertex set $\{1, 2, \ldots, v_G\}$ which contain a given subgraph H on the vertex set $\{1, 2, \ldots, v_H\}$, and are isomorphic to G. We will use the identity

$$f(H, G) \frac{v_G!}{aut(G)} = \binom{v_G}{v_H} \frac{v_H!}{aut(H)} c(H, G)$$

which can be verified as follows. There are $\frac{v_G!}{aut(G)}$ graphs on vertex set $1, 2, \ldots, v_G$ which are isomorphic to G, and there are $f(H, G)$ ways of choosing the subgraph H in G. On the other hand, one can choose first the vertex set of the graph H (in $\binom{v_G}{v_H}$ ways), build a copy of H on that set (in $\frac{v_H!}{aut(H)}$ ways), and finally extend H to G (in $c(H, G)$ ways).

Using the above identity we obtain

$$\begin{aligned} ES_H &= f(H, G) \binom{n - v_G}{v_G - v_H} c(H, G) p^{l_G - l_H} \sim \\ &\sim [f(H, G)]^2 \frac{aut(H)}{aut(G)} n^{v_G - v_H} p^{l_G - l_H} = s n^{v_G - v_H} p^{l_G - l_H} \ , \end{aligned} \tag{1.10}$$

where s is defined in Theorem 2. If $\omega(n)$ tends to ∞ more slowly than $\log n$ does, then $p \asymp n^{-\frac{1}{\alpha}}(\log n)^{\frac{1}{l_G-l_H}}$ and from (1.9) and (1.10) we have

$$\begin{aligned} P(Z>0) &\le EZ \le \frac{n^{v_G}p^{l_G}}{aut(G)}\exp\{-sn^{v_G-v_H}p^{l_G-l_H}+o(1)\} = \\ &= O\Bigg(n^{v_G}n^{-\frac{l_G}{\alpha}}(\log n)^{\frac{l_G}{l_G-l_H}} \\ &\exp\Big\{-\Big[\frac{\alpha v_G-l_G}{\alpha}\log n+\frac{l_G}{l_G-l_H}\log\log n+s\omega(n)\Big]\Big\}\Bigg) = \\ &= O\left(n^{\frac{\alpha v_G-l_G}{\alpha}}(\log n)^{\frac{l_G}{l_G-l_H}}n^{-\frac{\alpha v_G-l_G}{\alpha}}(\log n)^{-\frac{l_G}{l_G-l_H}}e^{-s\omega(n)}\right) = \\ &= O\left(e^{-s\omega(n)}\right) = o(1). \end{aligned} \tag{1.11}$$

Otherwise we obtain

$$P(Z>0) \le O\left(n^{\frac{v_G\alpha-l_G}{\alpha}}(\omega)^{\frac{2l_G}{l_G-l_H}}\exp\{-s\omega(n)\}\right) = o(1).$$

Thus, in all three cases, $P(Z>0)=o(1)$. This completes the proof of 0-statement.

Let us now prove the 1-statement. First we show that if $\omega(n)\to-\infty$ and $np^{d_G}\to\infty$ then $EZ\to\infty$. By the FKG inequality,

$$P(S=0) \ge \prod_{K\subset G} P(S_K=0)\ .$$

Denote by $N(K,G)$ the number of copies of G which intersect G_0 on a subgraph isomorphic to K. Thus, $ES_K = N(K,G)p^{l_G-l_K}$.

Let us bound $P(S_K=0)$ from below. Again by the FKG inequality

$$\begin{aligned} P(S_K=0) &\ge \left(1-p^{l_G-l_K}\right)^{N(K,G)} \ge \\ &\ge \exp\Big\{-\frac{p^{l_G-l_K}N(K,G)}{1-p^{l_G-l_K}}\Big\} = \exp\Big\{-p^{l_G-l_K}N(K,G)+o(1)\Big\} = \\ &= \exp\{-ES_K+o(1)\}. \end{aligned}$$

Note that, since $\alpha < \frac{l_G-l_K}{v_G-v_K}$ for every $K\neq H,G$,

$$ES_K \asymp \frac{\Psi_G}{\Psi_K} \le n^{v_G-v_K-\frac{l_G-l_K}{\alpha}}(\log n)^{\frac{l_G-l_K}{l_G-l_H}} < n^{-\varepsilon'}, \tag{1.12}$$

where $\varepsilon' = \frac{1}{2\alpha}(l_G-l_K)-\frac{1}{2}(v_G-v_K)>0$.

Therefore

$$P(S=0) \ge \exp\{-ES_H+o(1)\}. \tag{1.13}$$

and

$$EZ \ge EX\exp\{-ES_H+o(1)\} \sim \frac{n^{v_G}p^{l_G}}{autG}\exp\{-sn^{v_G-v_H}p^{l_G-l_H}+o(1)\}\ .$$

It is easy to check that under the assumptions of the 1-statement the right hand side of the above inequality tends to ∞.

To prove the 1-statement we use the second moment method in the form of the inequality

$$P(Z=0) \le \frac{VarZ}{(EZ)^2} = \frac{E(Z(Z-1)) + EZ - (EZ)^2}{(EZ)^2}.$$

We shall show that

$$E(Z(Z-1)) \sim (EZ)^2 \ , \tag{1.14}$$

which, together with $EZ \to \infty$, will imply that

$$P(Z=0) \le o(1) + \frac{1}{EZ} = o(1).$$

Let $\mathcal{G}^{**}(n,p)$ be the random graph where the edges of two fixed and disjoint copies G' and G'' of G are present with probability 1 and the remaining edges are present, as usual, with probability p, independently from each other. Then

$$E(Z(Z-1)) \sim (EX)^2 P(S' = S'' = 0) \ ,$$

where S' (S'') denotes the number of copies of G in $\mathcal{G}^{**}(n,p)$ which are not vertex disjoint from G' (G''). Together with (1.1) this means that to show (1.14) it remains to prove that

$$[P(S=0)]^2 \sim P(S' = S'' = 0). \tag{1.15}$$

Applying the FKG inequality to the space $\mathcal{G}^{**}(n,p)$, one obtains

$$P(S' = S'' = 0) \ge P(S' = 0)P(S'' = 0) \ ,$$

where, let us recall, the random variable S, defined in the space $\mathcal{G}^{*}(n,p)$. counts copies of G not vertex disjoint from G_0. The asymptotic equation

$$P(S'=0) = P(S''=0) \sim P(S=0)$$

follows from the fact that every copy of G which intersects both G' and G'' shares with $G' \cup G''$ a disconnected subgraph K. Thus, $K \ne H, G$ and, by (1.12), the expected number of such copies in $\mathcal{G}^{**}(n,p)$ is $o(1)$.

Hence, by (1.13),

$$P(S' = S'' = 0) \ge (1 + o(1))[P(S=0)]^2 \ge \exp\{-2ES_H + o(1)\}. \tag{1.16}$$

For a given copy G_i of G such that $G_i \cap G' \cong H$ and $V(G_i) \cap V(G'') = \emptyset$ define the indicator variable

$$J_i' = \begin{cases} 1 & \text{if} \quad G_i \subset \mathcal{G}^{**}(n,p) \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for a copy G_i of G such that $G_i \cap G'' \cong H$ and $V(G_i) \cap V(G') = \emptyset$ define

$$J_i'' = \begin{cases} 1 & \text{if} \quad G_i \subset \mathcal{G}^{**}(n,p) \\ 0 & \text{otherwise.} \end{cases}$$

Let $S_H{'} = \sum_{G' \cap G_i \cong H} J_i{'}$, $S_H{''} = \sum_{G'' \cap G_i \cong H} J_i{''}$ and $\Delta_0 = \sum\sum E(J_i J_j)$, where the double summation is taken over all pairs (i,j), $i \neq j$, such that $G_i \cap G' = H$ or $G_i \cap G'' = H$, the same holds for G_j, and $(E(G_i) \cap E(G_j)) \setminus (E(G') \cup E(G'')) \neq \emptyset$.

By a weaker version of inequality (1.4) (see [3]) applied to $S_H{'} + S_H{''}$,

$$P(S' + S'' = 0) \leq P(S_H{'} + S_H{''} = 0) \leq$$
$$\leq \exp\{-E(S_H{'} + S_H{''}) + \Delta_0\} = \exp\{-2ES_H + o(1) + \Delta_0\}.$$

To complete our proof it remains to show that $\Delta_0 = o(1)$. We split $\Delta_0 = \Delta_1 + \Delta_2$, where $\Delta_1 = \sum\sum E(J_i J_j)$, the summation over all pairs (i,j), $i \neq j$, such that G_i and G_j intersect $G' \cup G''$ on the same copy of H, and $\Delta_2 = \sum\sum E(J_i J_j)$, the summation over all pairs (i,j), $i \neq j$, such that G_i and G_j intersect $G' \cup G''$ on different (thus disjoint – cf. Fact) copies of H which may belong to the same or to different graphs. These two cases are illustrated in Figures 5 and 6.

Fig.5 Fig.6

Since $\frac{\Psi_G}{\Psi_H} = O(\log n)$, and for every $K \subset G$

$$\Psi_K = O((np^{d_K})^{v_{\tilde{H}}}) = O((np^{d_G} p^{-(d_G - d_K)})^{v_K}) \geq n^{\varepsilon''},$$

where $\varepsilon'' > \frac{v_K(d_G - d_K)}{\alpha}$, we have by (1.12)

$$\Delta_1 = O\left(\sum_{H \subset K \neq G} \frac{{\Psi_G}^2}{\Psi_H \Psi_K} \right) = O((\log n) n^{-\varepsilon'}) = o(1)$$

and

$$\Delta_2 = O\left(\sum_{H \subset K \neq G} \frac{{\Psi_G}^2}{{\Psi_H}^2 \Psi_K} \right) = O((\log n)^2 n^{-\varepsilon''}) = o(1),$$

where K represents the intersection of G_i and G_j outside $G' \cup G''$. This completes the proof of the 1-statement in Theorem 2.

The first part of the Poisson-statement is contained in Theorem 1. Let us show that Z is asymptotically Poisson distributed also at the second threshold. We will use the method of moments, i.e. will check whether all factorial moments of Z converge to the corresponding factorial moments of the Poisson distribution with the required expectation.

Fix an integer r, $r \geq 1$, and let $\mathcal{G}^{(r)}(n,p)$ denote the random graph in which the edges of given disjoint copies $G_1, G_2, \dots, G_r$ of G appear with probability 1 and the remaining edges with probability p independently of each other.

Let $S^{(r)}$ be the number of copies of G in $\mathcal{G}^{(r)}(n,p)$ which intersect at least one of $G_1, G_2, \dots, G_r$. Furthermore, for $K \subset G$, let $S_K^{(r)}$ denote the number of such copies which intersect $G_1 \cup G_2 \cup \ldots \cup G_r$ on a subgraph isomorphic to K.

Consider the r-th factorial moment of Z. We have for $r = 1, 2, \dots$

$$\begin{aligned} E((Z)_r) &= \binom{n}{v_G, v_G, \dots, n - r v_G} \left(\frac{v_G!}{autG}\right)^r p^{r l_G} P(S^{(r)} = 0) \\ &\sim (EX)^r P(S^{(r)} = 0) \ , \end{aligned} \tag{1.17}$$

Similarly as in the proof of the 1-statement, one can show, using the weak version of inequality (1.4) on one side and the FKG inequality and (1.12) on the other side, that

$$P(S^{(r)} = 0) = \exp\{-ES_H^{(r)} + o(1)\} \ .$$

It remains to estimate $ES_H^{(r)}$. Since H is connected, any copy of G which intersects $G_1 \cup G_2 \cup \ldots \cup G_r$ on a subgraph isomorphic to H intersects precisely one of the r copies. Hence,

$$ES_H^{(r)} = r f(H,G) \binom{n - r v_G}{v_G - v_H} c(H,G) p^{l_G - l_H} \sim r s n^{v_G - v_H} p^{l_G - l_H} .$$

Finally, from (1.17) we obtain

$$E((Z)_r) \to \left(\frac{1}{aut(G)} \left[\frac{\alpha v_G - l_G}{\alpha s} \right]^{\frac{l_G}{l_G - l_H}} e^{-sc} \right)^r \ ,$$

which completes the proof of Theorem 2. □

2.3. Proof of Theorem 4.

The following proof was suggested by T. Łuczak.

Fix an integer $v \geq 1$ and call a vertex of a graph *small* if it has less than $2v-1$ neighbors and *large* otherwise. Let $\mathcal{P}_v$ denote the property of a graph F that, for every $k \leq v+2$ no $v+1$ small vertices of F belong to a connected k-vertex subgraph of F. The following result constitutes the probabilstic ingredient of the proof of Theorem 4.

Lemma. *Let v be a positive integer and, for some $\varepsilon > 0$, let $np(v+1) > (1+\varepsilon)\log n$. Then aas $\mathcal{G}(n,p)$ possesses the property $\mathcal{P}_v$.*

Proof. Let Y_k be the number of k-vertex trees in $\mathcal{G}(n,p)$ with $v+1$ small vertices, $k = v+1, v+2$. Using the first moment method we have

$$P(Y_k > 0) \leq EY_k < \binom{n}{k} k^{k-2} p^{k-1} \binom{k}{v+1} \left[\sum_{t=0}^{2v-2} \binom{n-k}{t} p^t (1-p)^{n-k-t} \right]^{v+1} =$$
$$= O\left(n(np)^{k-1+(2v-2)(v+1)} e^{-np(v+1)} \right) = o(1) \ ,$$

if $np = \Theta(\log n)$. To complete the proof of the Lemma notice that if $np(v+1) > (k + (2v-2)(v+1) + \varepsilon) \log n$ then

$$P(Y_k > 0) \leq EY_k < O\left(n^{k+(2v-2)(v+1)} e^{-(k+(2v-2)(v+1)+\varepsilon)\log n} \right) = o(1)$$

as well. □

Obviously, for any graph G, if G' is an isolated copy of G in a graph then G' is also solitary . We shall now show a deterministic statement that if a graph F satisfies property $\mathcal{P}_v$ then every solitary copy of a v-vertex tree in F must be isolated. This statement implies Theorem 4, since when, say $\omega(n) \geq -\log\log n$ and $\varepsilon = \frac{1}{2v}$, then the assumption of the Lemma holds, and the known results on the existence and distribution of the number of isolated trees in $\mathcal{G}(n,p)$ (cf. [1]) apply. (When $\omega(n) \to -\infty$ at any rate then for a given $v-$ vertex tree *aas* there are isolated copies, and thus solitary copies of that tree in $\mathcal{G}(n,p)$.)

Let T be a v-vertex tree on vertices $x_1, ..., x_v$ ordered in such a way that x_1 is a pendant vertex, and for each $i = 2, ..., v$ there is an index $j = j(i) < i$ with $\{x_j, x_i\} \in E(T)$. Furthermore, let for each $i = 2, ..., v$, T_i be the subtree of T induced by the vertices $x_1, ..., x_i$. (Note that both x_1 and x_i are pendant vertices of T_i.)

Suppose that there is a graph F satisfying property $\mathcal{P}_v$ and containg a solitary copy T' of T which is not isolated. We will now prove by induction on i that for each $i = 2, ..., v$ there is in F a copy T_i' of T_i with vertices $x_1', ..., x_i'$ corresponding to the vertices $x_1, ..., x_v$ of T_i under an isomorphism, and such that

(1) $V(T_i') \cap V(T') \neq \emptyset$, and
(2) all vertices of T_i', with a posssible exception of x_1' are large.

Note that the case $i = v$ of the above statement yields a contradiction with the solitude of T'.

Consider first the case $i = 2$. Since T' is not isolated, there are vertices $x \in V(T')$ and $y \in V(F) \setminus V(T')$ such that $\{x, y\} \in E(F)$. By property $\mathcal{P}_v$ there is at least one large vertex among the $v+1$ vertices of the set $V(T') \cup \{y\}$. Pick as x'_1 and x'_2 two adjacent vertices of F such that $x'_1 \in V(T')$ and x'_2 is large. The edge $\{x'_1, x'_2\}$ is the required copy of T_2.

Assume now that, for some $i \geq 3$, there is a copy T'_{i-1} of T_{i-1} in F satisfying (1) and (2). Since $j = j(i) \geq 2$, the vertex x'_j, corresponding to x_j, is large in F. By property $\mathcal{P}_v$ $(k = v+2)$, among its at least $2v-1$ neighbors there are at most v small vertices. On the other hand there are at most $i-2 \leq v-2$ neighbors of x'_j which belong to T'_{i-1}. Thus there is at least one large neighbor z of x'_j which can play the role of x'_i. The tree obtained from T'_{i-1} by adding vertex z and the edge $\{x'_j, z\}$ makes for the required copy of T_i. $\square$

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